Atmosphere, Ocean and Climate Dynamics
Answers to Chapter 9

1. *Consider an ocean of uniform density* $\rho_{ref} = 1000 \text{ kg m}^{-3}$.

   (a) *From the hydrostatic relationship, determine the pressure at a depth of* $1\,\text{km}$ *and at* $5\,\text{km}$. *Express your answer in units of atmospheric surface pressure,* $p_s = 1000\,\text{mbar} = 10^5\,\text{Pa}$.

   Hydrostatic balance is
   $$\frac{\partial p}{\partial z} = -g\rho \ ,$$
   whence, integrating from depth d ($z = -d$) to the surface ($z = h$),
   $$p(-d) = p(h) + g\rho(d + h) \ ,$$
   where ρ is the constant density. Since at a depth of a km or so, $d + h \simeq d$ we may write:
   $$\begin{aligned} p(-1\,\text{km}) &= \left(1 + \frac{9.81\,\text{m}^2\,\text{s}^{-1} \times 10^3\,\text{kg}\,\text{m}^{-3} \times 10^3\,\text{m}}{10^5\,\text{Pa}}\right) p_s \simeq 99 p_s \\ p(-5\,\text{km}) &= \left(1 + \frac{9.81\,\text{m}^2\,\text{s}^{-1} \times 10^3\,\text{kg}\,\text{m}^{-3} \times 5 \times 10^3\,\text{m}}{10^5\,\text{Pa}}\right) p_s \simeq 491 p_s \end{aligned}$$

   (b) *Given that the heat content of an elementary mass* dm *of dry air at temperature* T *is* $c_p T\ dm$ *(where* c_p *is the specific heat of air at constant pressure), find a relationship for, and evaluate, the (vertically integrated) heat capacity (heat content per degree Kelvin) of the atmosphere per unit horizontal area. Find how deep an ocean would have to be in order to have the same heat capacity per unit horizontal area.*

If the heat content per unit mass is $c_p T\ dm$, the heat capacity per unit mass is $c_p\ dm$. Hence the heat content of a vertically integrated column, per unit horizontal area, is

$$\int_0^\infty c_p \rho\ dz = -\frac{c_p}{g}\int_0^\infty \frac{\partial p}{\partial z} dz = c_p \frac{p(0)}{g} \ ,$$

assuming hydrostatic balance, where $p(0)$ is surface pressure = 1000hPa = 10^5Pa, since $p(\infty) = 0$. Therefore the vertically integrated heat capacity of the atmosphere is

$$\frac{1004 \times 10^5}{9.81} = 1.02 \times 10^7 \mathrm{JK\,m^{-2}}.$$

The heat capacity per unit area of an ocean of depth D is $c_p \rho D$, where $c_p = 4187 \mathrm{JK\,kg^{-1}}$ and $\rho = 1000\,\mathrm{kg\ m^{-3}}$ of course are the values for water. For such an ocean to have the same heat capacity as the atmosphere,

$$c_p \rho D = 1.02 \times 10^7 \mathrm{JK^{-1}\,m^{-2}}$$

so

$$D = \frac{1.02 \times 10^7}{4187 \times 1000} = 2.44\,\mathrm{m}\ .$$

This is very shallow! So, a real ocean of depth (say) 4 km has $4000/2.44 = 1639$ times the heat capacity of the atmosphere.

2. *Assume that, in the mixed layer, mixing maintains a vertically uniform temperature. A heat flux of $25\,Wm^{-2}$ is applied at the ocean surface. Taking a reasonable representative value for the mixed layer depth, determine how long it takes for the mixed layer to warm up by $1^o C$. [Use density of water= $1000 kg\ m^{-3}$; specific heat of water = $4187 J$ $\mathrm{kg}^{-1} K^{-1}$.]*

   Adopt a representative value of $D = 100$m for the mixed layer depth. Then the heat capacity of the mixed layer per unit area is (where c is specific heat and ρ density)

$$\begin{aligned} H &= c\rho D \\ &= 4.19 \times 10^8 \mathrm{JK^{-1}m^{-2}}. \end{aligned}$$

   Therefore the warming rate, when heated by a heat flux of $25\mathrm{Wm^{-2}}$, is

$$\frac{dT}{dt} = \frac{25}{4.19 \times 10^8} \simeq 6.0 \times 10^{-8} \mathrm{Ks^{-1}},$$

   so the time taken to warm up by 1K is

$$\begin{aligned} \tau &= 1.7 \times 10^7 \mathrm{s} \\ &\simeq 0.5\ \mathrm{yr}. \end{aligned}$$

Hence, *e.g.*, one expects to see a significant time lag of ocean temperatures with respect to solar forcing (i.e., warmest water well after summer solstice) and, in places where the mixed layer is deep, relatively little seasonal variability of mixed layer temperatures. Where the mixed layer is shallower than 100 m, the time scale is comparable with or less than a season, so there will be a larger annual variation of upper ocean temperature.

3. *Consider an ocean of uniform density $\rho = 1000\ kg\ m^{-3}$. The ocean surface, which is flat in the longitudinal direction, slopes linearly with latitude from $h = 0.1m$ above mean sea level (MSL) at $40^o N$ to $h = 0.1m$ below MSL at $50^o N$. Using hydrostatic balance, find the pressure at depth d below MSL. Hence show that the latitudinal pressure gradient $\partial p/\partial y$ and the geostrophic flow are independent of depth. Determine the magnitude and direction of the geostrophic flow at $45^o N$.*

   Consider Fig. 1.

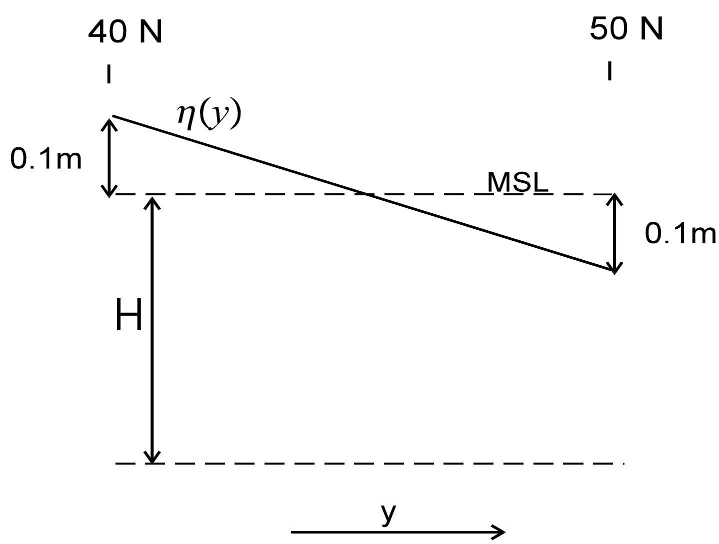


Figure 1:

Hydrostatic balance is

$$\frac{\partial p}{\partial z} = -g\rho\ ,$$

whence, integrating from depth d ($z = -d$) to the surface ($z = h(y)$),

$$p(-d) = p(h) + g\rho(d + h) \ ,$$

where ρ is the constant density. Pressure at the surface is atmospheric ($= p_s$) and constant, so

$$\frac{\partial p}{\partial y}(z = -d) = g\rho\frac{\partial h}{\partial y} \ , \tag{a}$$

which is independent of depth. This is because the mass per unit area above depth d varies horizontally only as a reflection of variations in surface height.

The geostrophic flow is zonal,

$$u = -\frac{1}{f\rho}\frac{\partial p}{\partial y}$$

or, using result (a),

$$u = -\frac{g}{f}\frac{\partial h}{\partial y} \ .$$

With the given data, $\partial h/\partial y = -0.2/\left(1.11 \times 10^6\right) = -1.8018 \times 10^{-7}$. Since $g = 9.81\text{ms}^{-2}$, and $f = 1.03 \times 10^{-4}\text{s}^{-1}$ at 45^{o}N,

$$u = \frac{9.81}{1.03 \times 10^{-4}} \times 1.8018 \times 10^{-7} = 1.72 \times 10^{-2}\text{ms}^{-1}$$

at 45^{o}N.

4. *Consider a straight, parallel, oceanic current at 45°N. For convenience, we define the x-and y- directions to be along and across the current, respectively. In the region $-L < y < L$, the flow velocity is*

$$u = U_0 \cos\left(\frac{\pi y}{2L}\right)\exp(\frac{z}{d})$$

where z is height (note that $z = 0$ at mean sea level and decreases downwards), $L = 100\,\text{km}$, $d = 400\,\text{m}$, and $U_0 = 1.5\,\text{m}\,\text{s}^{-1}$. In the region $|y| > L$, $u = 0$.

The surface current is plotted in the following figure:

Making use of the geostrophic, hydrostatic and thermal wind relations:

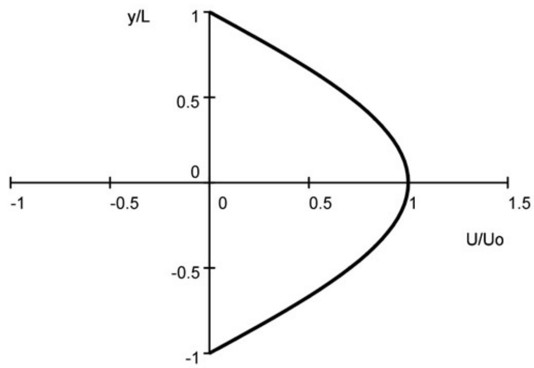


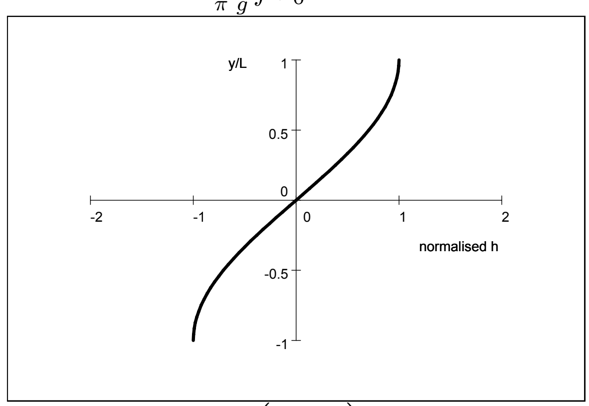
Figure 2: A plot of $\frac{u}{U_0}$ at the surface, $z = 0$, against $\frac{y}{L}$.

(a) *Determine and sketch the profile of surface elevation as a function of y across the current.*

Geostrophic balance is $fu + \frac{1}{\rho}\frac{\partial p}{\partial y} = 0$ and so at the surface $\frac{1}{\rho}\frac{\partial p}{\partial y} = g\frac{\partial h}{\partial y} = fU_0 \cos\left(\frac{\pi y}{2L}\right)$ implying that:

$$h = \int \frac{U_0}{g} \cos\left(\frac{\pi y}{2L}\right) dy = \frac{2}{\pi}\frac{L}{g} f U_0 \sin\left(\frac{\pi y}{2L}\right)$$

Inserting numbers we find that $\frac{2}{\pi}\frac{L}{g} f U_0 \simeq 1\,\mathrm{m}$.



A plot of $h/\left(\frac{2}{\pi}\frac{L}{g}U_0\right)$ against $\frac{y}{L}$.

(b) *Determine and sketch the density difference, $\rho(y, z) - \rho(0, z)$.*

The thermal wind equation is: $f\frac{\partial u}{\partial z} = \frac{g}{\rho_{ref}}\frac{\partial \rho}{\partial y}$and so $\frac{\partial \rho}{\partial y} = \frac{f\rho_{ref}}{g}\frac{\partial u}{\partial z} = \frac{f\rho_{ref}}{gd}U_0 \cos\left(\frac{\pi y}{2L}\right)\exp(\frac{z}{d})$. Integrating wrt y, we find:
$\rho(y,z) - \rho(0,z) = \frac{f\rho_{ref}}{gd}U_0 \exp(\frac{z}{d})\int \cos\left(\frac{\pi y}{2L}\right)dy = \frac{2L}{\pi}\frac{f\rho_{ref}}{gd}U_0\exp(\frac{z}{d})\sin\left(\frac{\pi y}{2L}\right)$ and so varies like h but decays exponentially with depth.

(c) *Assuming the density is related to temperature by Eq.(4.4) determine the temperature difference, $T(L,z) - T(-L,z)$, as a function of z. Evaluate this difference at a depth of 500 m. Compare with Fig.9.21.*
Since $\frac{\rho(y,z)-\rho(0,z)}{\alpha\rho_{ref}} = -\left(T(y,z) - T(0,z)\right) = -\frac{2L}{\pi}\frac{f}{\alpha g d}U_0\exp(\frac{z}{d})\sin\left(\frac{\pi y}{2L}\right)$, at a depth of 500 m we find a temperature change across the channel of magnitude $2 \times \left(\frac{2L}{\pi}\frac{f}{\alpha g d}U_0\right)\exp(\frac{-500}{400}) \sim 7\,^\circ\mathrm{C}$, roughly in accord with Fig.9.21.

5. *Figure 9.21 is a cross-section (in a plane normal to the flow) across the Gulf Stream at 38° N.*

The figure shows the distribution of temperature as a function of depth and of horizontal distance across the flow. Assume for the purposes of this question (all parts) that the flow is geostrophic.

(a) *Using hydrostatic balance, and assuming that atmospheric pressure is uniform and that horizontal pressure gradients vanish in the deep ocean, estimate the differences in surface elevation across the Gulf Stream (i.e., between the positions marked 73° W and 70° W on the figure). Neglect the effect of salinity on density, and assume that the dependence of density ρ on temperature T is adequately described by*

$$\rho = \rho_0\left[1 - \alpha\left(T - T_0\right)\right] \ ,$$

where ρ_0 is the density at 0° C, and α is the coefficient of thermal expansion.

Hydrostatic balance gives

$$\frac{\partial p}{\partial z} = -g\rho = -g\rho_0[1 - \alpha(T - T_0)],$$

so, if the surface elevation (above MSL $z = 0$) is $h(x)$, and assuming $p = p_a$ =constant at the surface

$$\begin{aligned} p(x,z) &= p_a + g\rho_0 \int_z^h [1 - \alpha(T - T_0)]\, dz \\ &\simeq p_a + g\rho_0(h - z) - g\rho_0\alpha \int_z^h (T - T_0)\, dz\ , \end{aligned}$$

assuming density variations are small. Given no horizontal pressure gradient in the deep ocean, $z = -Z_0$, say,

$$\delta p = p(x + \delta x, -Z_0) - p(x, -Z_0) = g\rho_0 \delta h - g\rho_0 \alpha \int_{-Z_0}^{z} \delta T\, dz\ .$$

So the elevation difference is

$$\delta h = \alpha \int_{-Z_0}^{z} \delta T\, dz\ .$$

From the figure, I estimate a mean δT between 73^o W and 70^o W of about 5K, averaged over the layer above 1000m depth, and $\delta T = 0$ beneath 1000m. Therefore

$$\int_{-Z_0}^{z} \delta T\, dz \simeq 5000 \text{ Km}$$

and so, using $\alpha = 2 \times 10^{-4}\text{K}^{-1}$,

$$\delta h = 1.0\text{m}\ .$$

(b) *The near-surface geostrophic flow u is related to surface elevation η by*

$$\mathbf{u} = \frac{g}{f}\widehat{\mathbf{z}} \times \nabla\eta\ ,$$

where g is gravity and f the Coriolis parameter. Explain how this equation is consistent with the geostrophic relationship between Coriolis force and pressure gradient.

See Section 9.3.1, pg.184.

(c) *Assuming (for simplicity) that the Gulf Stream's velocity is uniform down to a depth $D = 500m$, and that it is zero below this depth, use the near-surface geostrophic relationship*

$$\mathbf{u} = \frac{g}{f}\widehat{\mathbf{z}} \times \nabla h$$

to show that the net (integrated) water transport (i.e., the volume flux) along the Gulf Stream at this latitude is

$$\mathcal{V} = \frac{gD}{f}\ \delta h\ ,$$

where δh is the elevation difference you estimated in part (a). Evaluate this transport in units of Sverdrups (Sv). [$1Sv = 10^6 m^3 s^{-1}$; this is a conventional unit of water flow in oceanography.]

The $y-$velocity (along the stream) is

$$v = \frac{g}{f}\frac{\partial h}{\partial x}\ .$$

Integrating across the stream, the net volume flux along the Gulf Stream is

$$\begin{aligned} \mathcal{V} &= \int v\ d(area) \\ &= \frac{g}{f}\iint \frac{\partial h}{\partial x}\ dx\ dz \end{aligned}$$

where the integral is across the whole region of nonzero velocities. Hence

$$\begin{aligned} \mathcal{V} &= \frac{g}{f}\int_{-D}^{0}\int_{x_1}^{x_2} \frac{\partial h}{\partial x}\ dx\ dz\ , \\ &= \frac{gD}{f}\delta h \end{aligned}$$

where $\delta h = h(x_2) - h(x_1)$ is the height difference across the stream. Putting in the numbers, noting that $f(38\mathrm{N}) = 8.96 \times 10^{-5}\mathrm{s}^{-1}$

$$\begin{aligned} \mathcal{V} &= \frac{9.81 \times 500}{8.96 \times 10^{-5}} \times 1.0 = 5.47 \times 10^7 \mathrm{m}^3\mathrm{s}^{-1} \\ &= 54.7\ \mathrm{Sv}. \end{aligned}$$

6. *Fig.3 shows the trajectory of a 'champion' surface drifter which made one and a half loops around Antarctica between March, 1995 and March, 2000 (courtesy of Nikolai Maximenko). Red dots mark the position of the float at 30 day interval.*

   (a) *Compute the mean speed of the drifter over the 5-year period.*

   The mean speed of the drifter is $\frac{22\times10^3\,\mathrm{km}\times1\frac{1}{2}}{5\,\mathrm{y}} = 0.2\,\mathrm{m\,s^{-1}}$, where we have used the fact that it is $22,000\,\mathrm{km}$ around the path of the ACC.

   (b) *Assuming that the mean zonal current at the bottom of the ocean is zero, use the thermal wind relation (neglecting salinity effects) to compute the depth-averaged temperature gradient across the Antarctic Circumpolar Current. Hence estimate the mean temperature drop across the* $600\,\mathrm{km}$*-wide Drake Passage.*

   The thermal wind equation is: $f\frac{\partial u}{\partial z} = \frac{g}{\rho_{ref}}\frac{\partial \rho}{\partial y} = g\alpha\frac{\partial T}{\partial y}$. Thus: $\displaystyle\int \frac{\partial T}{\partial y}dz = \frac{f}{g\alpha}\int\frac{\partial u}{\partial z}dz = \frac{f}{g\alpha}\left(u_{\mathrm{surface}} - u_{\mathrm{bottom}}\right) = \frac{f}{g\alpha}u_{\mathrm{surface}}$. The depth averaged temperature drop is thus: $\frac{f}{g\alpha}u_{\mathrm{surface}}\frac{L_y}{H}$ which, inserting numbers, yields $\frac{10^{-4}\,\mathrm{s^{-1}}}{9.81\,\mathrm{m\,s^{-2}}\times2\times10^{-4}\,\mathrm{K^{-1}}}0.2\,\mathrm{m\,s^{-1}}\frac{600\,\mathrm{km}}{4\,\mathrm{km}} = 1.5\,\mathrm{K}$.

   (c) *If the zonal current of the ACC increases linearly from zero at the bottom of the ocean to a maximum at the surface (as measured by the drifter), estimate the zonal transport of the ACC through Drake Passage assuming a meridional velocity profile as in Fig.2 and that the depth of the ocean is* $4\,\mathrm{km}$. *The observed transport through Drake Passage is* $130Sv$. *Is your estimate roughly in accord? If not, why not?*

   Integrating $u = U_0\cos\left(\frac{\pi y}{2L}\right)\left(1-\frac{z}{H}\right)$ over the channel and over depth we obtain a transport of $\frac{2}{\pi}U_0LH$ which, inserting numbers yields: $\frac{2}{\pi}0.2\,\mathrm{m\,s^{-1}}\times300\,\mathrm{km}\times4\,\mathrm{km} = 1.5279\times10^8\,\mathrm{m^3\,s^{-1}} = 153\mathrm{Sv}$. This is not far from the observed volume transport of the ACC, which is 135Sv or so. Note that here we have only computed that part of the transport associated with the thermal wind — we must add on to this the transport associated with the depth independent (bottom) current which is 7.6Sv for every $1\,\mathrm{cm\,s^{-1}}$of bottom current.

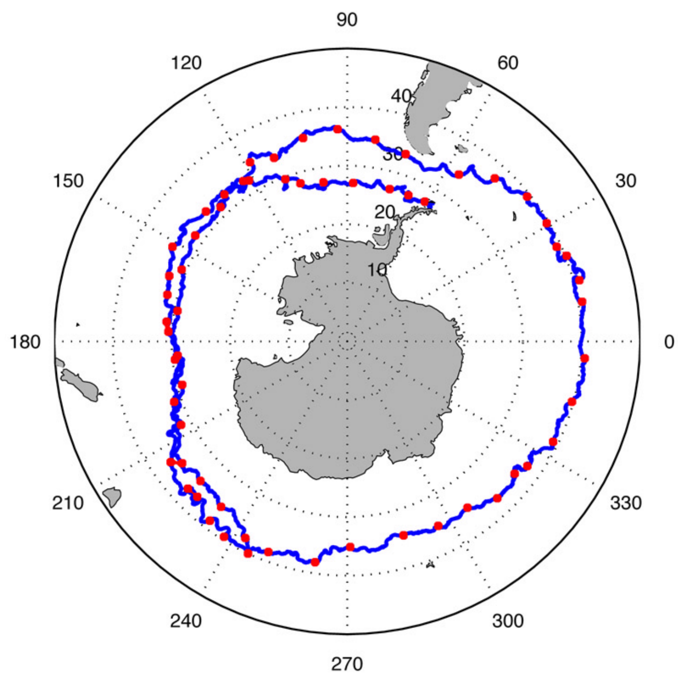


Figure 3: The trajectory of a surface drifter which made one and a half loops around Antarctica between March, 1995 and March, 2000 (courtesy of Nikolai Maximenko). Red dots mark the position of the float at 30 day intervals.