Atmosphere, Ocean and Climate Dynamics
Answers to Chapter 8

1. *Consider a zonally symmetric circulation (i.e., one with no longitudinal variations) in the atmosphere. In the inviscid upper troposphere, one expects such a flow to conserve absolute angular momentum, i.e., $DA/Dt = 0$, where $A = \Omega a^2 \cos^2\varphi + ua\cos\varphi$ is the absolute angular momentum per unit mass — see Eq.(8.1) — where Ω is the Earth rotation rate, u the eastward wind component, a the Earth's radius, and φ latitude.*

   (a) *Show, for inviscid zonally symmetric flow, that the relation $DA/Dt = 0$ is consistent with the zonal component of the equation of motion (using our standard notation, with F_x the x-component of the friction force per unit mass)*

   $$\frac{Du}{Dt} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \mathcal{F}_x \ ,$$

   in (x, y, z) coordinates, where $y = a\varphi$ (see Fig.6.19).

   For inviscid axisymmetric flow, conservation of angular momentum $DA/Dt = 0$ implies

   $$\frac{D}{Dt}\left(ua\cos\varphi + \Omega a^2\cos^2\varphi\right) = \left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)\left(ua\cos\varphi + \Omega a^2\cos^2\varphi\right) = 0\,.$$

   (Here $dx = a\cos\varphi\ d\lambda$ and $dy = a\ d\varphi$.) Since the planetary term is independent of x, z, and t,

   $$\begin{aligned}\frac{D}{Dt}\left(\Omega a^2\cos^2\varphi\right) &= v\frac{\partial}{\partial y}\left(\Omega a^2\cos^2\varphi\right)\\ &= \Omega a v\frac{\partial}{\partial\varphi}\cos^2\varphi\\ &= -2\Omega a v\sin\varphi\cos\varphi\\ &= -fv\times a\cos\varphi\ .\end{aligned}$$

   Similarly,

   $$\frac{D}{Dt}\left(ua\cos\varphi\right) = a\cos\varphi\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)u + uv\frac{\partial}{\partial\varphi}\cos\varphi = a\cos\varphi\left(\frac{Du}{Dt}\right)$$

where, for the vector component u,

$$\frac{Du}{Dt} = \left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right) u - \frac{uv}{a}\tan\varphi \ .$$

(The last term is a "metric" term, associated with the convergence of meridians, that has no counterpart in planar geometry.) Hence, adding the two terms and dividing by $a\cos\varphi$, we arrive at

$$\frac{Du}{Dt} - fv = 0 \ ,$$

which obviously corresponds with the zonal equation of motion in the case of inviscid ($\mathcal{F}_x = 0$) and axisymmetric ($\partial p/\partial x = 0$) flow.

(b) *Use angular momentum conservation to describe how the existence of the Hadley circulation explains the existence of both the subtropical jet in the upper troposphere and the near-surface Trade Winds.*

In the upper troposphere, the flow leaves the rising branch of the Hadley cell at the equator $\varphi = 0$ (*cf.* Fig. 8.5) with angular momentum density $A_0 = \Omega a^2$, if we assume that the flow rises from the ground there with no relative motion. If angular momentum is conserved in the upper level outflow (which we thus assume to be inviscid and axisymmetric) then at latitude φ, we have

$$A = ua\cos\varphi + \Omega a^2\cos^2\varphi = A_0 = \Omega a^2 \ ,$$

so $ua\cos\varphi = \Omega a^2(1-\cos^2\varphi)$, *i.e.*,

$$u = \Omega a\frac{\sin^2\varphi}{\cos\varphi} \ . \tag{1}$$

Thus, the winds must become westerly in such a way that the relative angular momentum term $ua\cos\varphi$ compensates for the decrease of the planetary contribution $\Omega a^2\cos^2\varphi$ as the air moves away from the equator. The zonal flow will be greatest at the edge of the cell, where φ is greatest, thus producing the subtropical jet. If the return flow, in the lower troposphere, were inviscid and thus also conserved angular momentum with $A_{low} = A_0$, then at a given latitude the low level flow would be the same as that aloft, since in Eq.(1) u is a function of φ only. However, in reality this low-level flow is under the influence of surface friction and A will therefore be progressively reduced. Thus,

$$A_{low} = u_{low} a \cos\varphi + \Omega a^2 \cos^2\varphi < \Omega a^2$$

whence

$$u_{low} < \Omega a \frac{\sin^2\varphi}{\cos\varphi} .$$

At some point (before $\varphi \to 0$) u_{low} will become negative, and so the low level flow will be equatorward and eastward there. This is the region of the Trade winds.

(c) *If the Hadley circulation is symmetric about the equator, and its edge is at* 20° *latitude, determine the strength of the subtropical jet.*

Using the equation derived in part (a), the predicted zonal wind at 20^o latitude is

$$u = 7.27 \times 10^{-5} \times 6.37 \times 10^6 \times \frac{\sin^2(20^o)}{\cos(20^o)} = 57.6 \ \mathrm{m\,s^{-1}} .$$

(The observed zonal winds are weaker than this. In reality, non-axisymmetric atmospheric eddies act to reduce angular momentum in the outflow, and hence reduce the strength of the jets.)

2. *Consider the tropical Hadley circulation in northern winter, as shown schematically in Fig.1. The circulation rises at* $10^\circ S$*, moves northward across the equator in the upper troposphere, and sinks at* $20^\circ N$*. Assuming that the circulation, outside the near-surface boundary layer, is zonally symmetric (independent of* x*) and inviscid (and thus conserves absolute angular momentum about the Earth's rotation axis), and that it leaves the boundary layer at* $10^\circ S$ *with zonal velocity* $u = 0$*, calculate the zonal wind in the upper troposphere at (a) the equator, (b) at* $10^\circ N$*, and (c) at* $20^\circ N$*.*

The angular momentum density (in our usual notation) is

$$M = \Omega a^2 \cos^2\varphi + u a \cos\varphi .$$

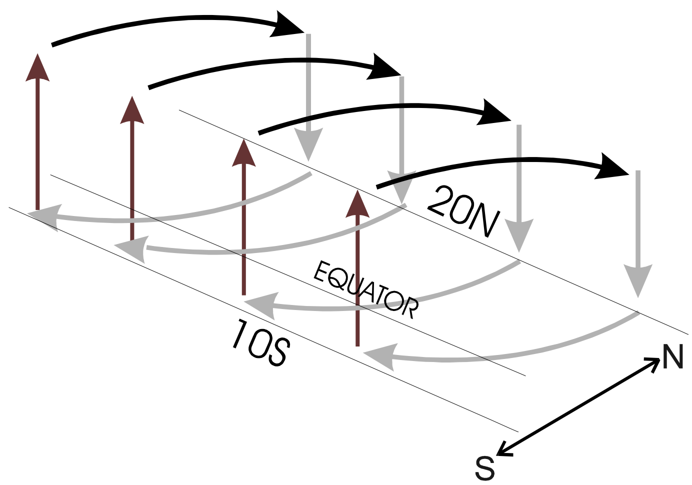


Figure 1:

In the upper tropospheric branch of the circulation, angular momentum is conserved, so $M = M_0$ is constant. Note that $\Omega a^2 = \left(\frac{2\times\pi}{86400}\right) \times \left(6.371 \times 10^6\right)^2 = 2.952 \times 10^9 \mathrm{m}^2\mathrm{s}^{-1}$. Since at 10°S, $u = 0$,

$$\begin{aligned} M_0 &= \Omega a^2 \cos^2\left(10^\circ\right) = \Omega a^2 \cos^2\left(\frac{\pi}{18}\right) \\ &= 2.952 \times 10^9 \times \cos^2\left(\frac{\pi}{18}\right) = 2.863 \times 10^9 \mathrm{m}^2\mathrm{s}^{-1}. \end{aligned}$$

Therefore, at other latitudes,

$$u = \frac{M_0 - \Omega a^2 \cos^2\varphi}{a\cos\varphi} .$$

(a) At $\varphi = 0°$,

$$u = \frac{M_0 - \Omega a^2}{a} = -\frac{8.9 \times 10^7}{6.371 \times 10^6} = -13.9 m s^{-1}$$

Note that this is easterly—air moves further away from the rotation axis as it goes from 10°S to the equator.
(b) At $\varphi = +10°$,

$$u = \frac{M_0 - \Omega a^2 \cos^2\left(\frac{\pi}{18}\right)}{a\cos\left(\frac{\pi}{18}\right)} = 0 ,$$

since the air has now moved, after crossing the equator, back to the same distance from the rotation axis as when it started.

(c) At $\varphi = +20°$,

$$\begin{aligned} u &= \frac{2.863 \times 10^9 - 2.952 \times 10^9 \times \cos^2\left(\frac{\pi}{9}\right)}{6.371 \times 10^6 \times \cos\left(\frac{\pi}{9}\right)} \\ &= 42.8\mathrm{ms}^{-1}. \end{aligned}$$

This is westerly—the air has moved closer to the rotation axis and has therefore"spun up".

The profile of $u(\varphi)$ in the upper troposphere is in fact as shown in Fig. 2.

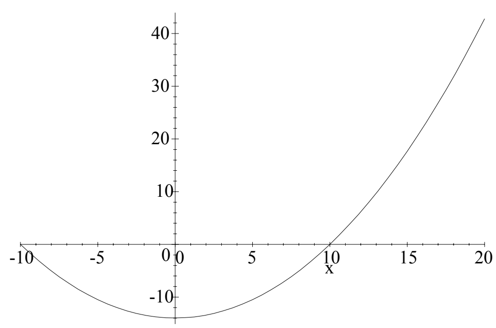


Figure 2: Plot of u (ms^{-1}) vs latitude (degrees).

3. *Consider what would happen if a force toward the pole were applied to the ring of air considered in Q.1 if it conserved its absolute angular momentum, A. Calculate the implied relationship between a small displacement $\delta\varphi$ and the change in the speed of the ring δu. How many kilometers northwards does the ring have to be displaced in order to change its relative velocity $10\,\mathrm{m\,s^{-1}}$? How does your answer depend on the equilibrium latitude? Comment on your result.*

   The angular momentum is (in standard notation):

$$A = \Omega a^2 \cos^2\varphi + ua\cos\varphi = A(u, \varphi) \tag{2}$$

Thus:

$$\delta A = \frac{\partial A}{\partial u} du + \frac{\partial A}{\partial \varphi} d\varphi = 0$$

if angular momentum is conserved. We see then that (computing the gradients of A wrt to φ and u from Eq.(2):

$$\frac{du}{d\varphi} = -\frac{\frac{\partial A}{\partial \varphi}}{\frac{\partial A}{\partial u}} = 2\Omega a \sin\varphi + u \tan\varphi = 2\Omega a \sin\varphi \left(1 + \frac{u}{\cos\varphi}\right)$$

If u were zero initially, $\delta u = 2\Omega a \sin\varphi \delta\varphi = f dy$, where f is the Coriolis parameter and $dy = a d\varphi$. Hence we obtain the simple result:

$$\delta u = f \delta y$$

Putting in numbers, we find that: $\delta y = \frac{10 m/s}{10^{-4}/s} = 100 km$, for a $\delta u = 10 m/s$. Thus we see that one only has to shift a ring $100 km$ to set up a strong zonal wind, that will quickly come in to geostrophic balance with the pressure gradient force.

4. *An open dish of water is rotating about a vertical axis at 1 revolution per minute. Given that the water is* $1\,^\circ\mathrm{C}$ *warmer at the edges than at the center at all depths, estimate, under stated assumptions and using the data below, typical azimuthal flow speeds at the free surface relative to the dish. Comment on, and give a physical explanation for, the sign of the flow.*

   How much does the free surface deviate from its solid body rotation form?

   Briefly discuss ways in which this rotating dish experiment is a useful analogue of the general circulation of the Earth's atmosphere.

   Assume the equation of state given by Eq.(4.4) with $\rho_{ref} = 1000\,\mathrm{kg\,m^{-3}}$, $\alpha = 2 \times 10^{-4}\,\mathrm{K^{-1}}$ *and* $T_{ref} = 15\,^\circ\mathrm{C}$, *the mean temperature of the water in the dish. The dish has a radius of* $10\,\mathrm{cm}$ *and is filled to a depth of* $5\,\mathrm{cm}$.

   Applying the thermal wind equation (7.18), the vertical shear in the geostrophic azimuthal flow is

$$\frac{\partial v_g}{\partial z} = \frac{\alpha g}{f} \frac{\partial T}{\partial r} \ .$$

For the stated problem, $f = 2\,(2\pi/60) = 0.21\ \mathrm{s}^{-1}$ (if we assume the rotation to be anticlockwise, so that f is positive), and $\partial T/\partial r \simeq \delta T/R$ where $\delta T = 1\,^{\circ}\mathrm{C}$ is the temperature difference between edge and center, and the tank radius is $R = 0.1\,\mathrm{m}$. Hence

$$\frac{\partial v_g}{\partial z} \simeq \frac{2\times 10^{-4}\times 9.81}{0.21} \times \frac{1}{0.1} = 0.093\ \mathrm{s}^{-1}\ .$$

Friction at the base of the tank will ensure that $v_g \simeq 0$ there; hence the flow velocity at the mean top ($z = h_0 = 0.05\,\mathrm{m}$) will be $v_g \simeq 0.093\times 0.05 = 0.0047\ \mathrm{m\,s}^{-1}$. The flow is anticyclonic (*i.e.*, "westward," with the same sense as the tank's rotation). A cross-section running from one edge of the tank to the other, across the center, is thus like that shown in the schematic Fig. 7.19, with a change of sign because in the present case the temperature increases outward, rather than decreasing poleward as in the atmosphere. At the surface, the vanishing of the geostrophic flow implies no horizontal gradient of excess pressure $p' = p - \rho_{ref}\Omega^2 r^2/2$ (*i.e.* the pressure field beyond that required to balance the centrifugal force); then hydrostatic balance dictates that pressure decreases more rapidly with height in the center of the tank than at the warmer, and more dense, edges. Hence the excess pressure increases with radius at any height above the surface and, consequently, the geostrophic flow is directed anticyclonically, with low pressure to its left for $f > 0$ (*cf.* Buys-Ballot's law) and to its right if $f < 0$.
At the surface, $z = h$, the geostrophic flow is

$$v_g = \frac{\alpha g}{f} h \frac{\partial T}{\partial r} = \frac{g}{f}\frac{\partial h}{\partial r}\ .$$

Hence, integrating, the (small) height difference δh between the edge and the center (in addition to that corresponding to the centrifugal term) is

$$\delta h = \alpha h_0 \delta T = 2\times 10^{-4}\times 0.05\times 1 = 1\times 10^{-5}\ \mathrm{m}\ ,$$

a hardly perceptible difference. (Note that this height difference is nothing more than that produced by thermal expansion, given the temperature differences between the fluid at the center of the tank at the edge.)

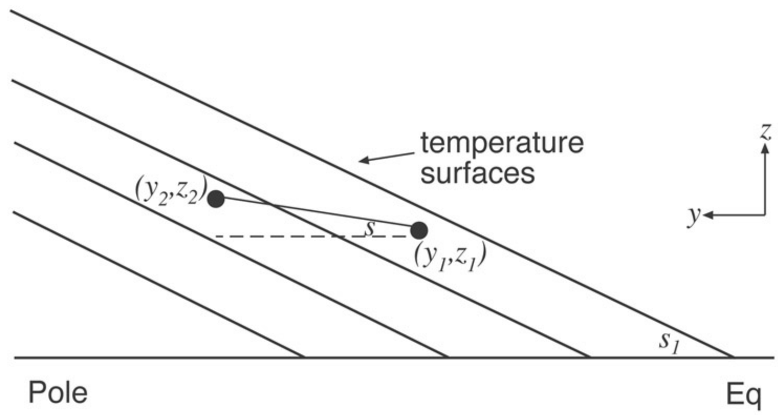


Figure 3: Schematic for energetic analysis of the thermal wind considered in Q.5. Temperature surfaces have slope s_1: parcels of fluid are exchanged along surfaces which have a slope s.

5. *Consider the incompressible, baroclinic fluid ($\rho = \rho(T)$) sketched in Fig.8.17 in which temperature surfaces slope upward toward the pole at an angle s_1. Describe the attendant zonal wind field assuming it is in thermal wind balance.*

   By computing the potential energy before and after interchange of two rings of fluid (coincident with latitude circles y at height z) along a surface of slope s, show that the change in potential energy $\Delta PE = PE_{final} - PE_{initial}$ is given by

$$\Delta PE = \rho_{ref} N^2 \left(y_2 - y_1\right)^2 s \left(s - s_1\right),$$

   where $N^2 = -\frac{g}{\rho_{ref}} \frac{\partial \rho}{\partial z}$ is the buoyancy frequency (see Section 4.4), ρ_{ref} is the reference density of the fluid and y_1, y_2 are the latitudes of the interchanged rings. You will find it useful to review Section 4.2.3.

   The change in potential energy is given by Eq.(4.7) which can be written, expressing it in terms of the buoyancy, $b = g \frac{\delta \rho}{\rho_{ref}}$:

$$\Delta PE = \rho_{ref} \left(z_2 - z_1\right) \left(b_2 - b_1\right) \tag{3}$$

   We also have $\Delta b = \frac{\partial b}{\partial y} \Delta y + \frac{\partial b}{\partial z} \Delta z$ so that

$$\left(b_2 - b_1\right) = M^2 \left(y_2 - y_1\right) + N^2 \left(z_2 - z_1\right)$$

The slope of the buoyancy surfaces are, setting $b_2 = b_1$, $s_1 = \frac{dy}{dz} = -\frac{M^2}{N^2}$. Hence if $s = \frac{(z_2-z_1)}{(y_2-y_1)}$ is the slope along which the parcels are exchanged, we may write:

$$(b_2 - b_1) = N^2 (y_2 - y_1) (s - s_1)$$

enabling us to express Eq.(3) thus:

$$\Delta PE = \rho_{ref} N^2 (y_2 - y_1)^2 \, s \, (s - s_1) ,$$

Hence show that for a given meridional exchange distance $(y_2 - y_1)$*:*

(a) *energy is released if* $s < s_1$
The sign of ΔPE is the same as the sign of the factor $s (s - s_1)$ and it will be negative, corresponding to the possibility of instability, if $s < s_1$.

(b) *the energy released is a maximum when the exchange occurs along surfaces inclined at half the slope of the temperature surfaces. This is the 'wedge of instability' discussed in Section 8.3.3 and illustrated in Fig.8.10.*

For a given exchange distance, $(y_2 - y_1)$, $-\Delta PE$ will be a maximum when $s (s - s_1)$ is a maximum, i.e. when $s = \frac{s_1}{2}$.

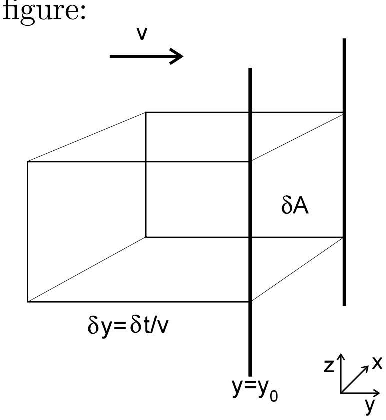
6. *Discuss, qualitatively but from basic principles, why most of the Earth's desert regions are found at latitudes of* $20 - 30°$*.*

Because of the equator-to-pole temperature gradient in the atmosphere, atmospheric isentropes (surfaces of constant potential temperature) slope upward toward the poles; there is thus a reservoir of available potential energy that can drive motions. In the tropics where the Coriolis parameter f is weak, the atmosphere behaves qualitatively like a nonrotating atmosphere would: it overturns, with rising motion where the temperatures are warmest (at the equator, on the annual average). However, the poleward flow in the upper troposphere moves into a region of increasing f, and the effects of rotation become apparent, producing a westerly component to the flow. In fact, this flow (if we assume it to be longitudinally uniform) conserves its absolute angular momentum (relative to an inertial reference frame). If the poleward flow were to extend all the way to the pole (so that its distance from the rotation axis $\to 0$) the westerly component of the flow would become infinite, which is unphysical. Thus, the flow—the "Hadley circulation"—must terminate before reaching the poles. In practice, this happens at a latitude between 20 and 30°. At this poleward edge of the circulation, the flow descends and returns to the equator in the lower troposphere. The descending flow at the poleward edge is dry (nearly all the moisture having been rained out in the rising branch) and warm (because of adiabatic compression). Thus, there is little moisture available for rain, and the warm overlying atmosphere produces a very stable layer just above the surface (the Trade Wind inversion), making convection difficult to achieve. Therefore, there is little rain in these regions, which form the desert belt in the subtropics.

7. *Given that the heat content of an elementary mass dm of air at temperature T is $c_pT\ dm$ (where c_p is the specific heat of air at constant pressure), and that its northward velocity is v:*

   (a) *show that the northward flux of heat crossing unit area (in the $x - z$ plane) per unit time is $\rho c_p vT$;*

   Consider the following figure:



   In a time δt, the volume of fluid crossing area δA located at $y = y_0$ in the $x - z$ plane is $\delta y\ \delta A$, where $\delta y = v\ \delta t$, and its mass is $\delta m = \rho\,\delta y\,\delta A$. The heat content of this volume is $c_pT \times \delta m = \rho c_pT \times \delta y \times \delta A = \rho c_pTv\ \delta t\ \delta A$. Therefore the flux of heat crossing y_0 per unit area per unit time is

$$\delta\mathcal{H} = \rho c_p T v.$$

(b) *hence, using the hydrostatic relationship, show that the net northward heat flux H in the atmosphere, at any given latitude, can be written*

$$\mathcal{H} = c_p \int_{x_1}^{x_2} \int_0^{\infty} \rho\, vT\, dx\, dz = \frac{c_p}{g} \int_{x_1}^{x_2} \int_0^{p_s} vT\, dx\, dp\ ,$$

where the first integral (in x) is completely around a latitude circle and p_s is surface pressure.

Integrating $\delta\mathcal{H} = \rho c_p T v$, the net heat flux crossing latitude y_0 is just

$$\mathcal{H} = \int \delta\mathcal{H}\, \delta A = c_p \int_{x_1}^{x_2} \int_0^{\infty} \rho\, vT\, dx\, dz\ ,$$

where $x_2 - x_1$ is the distance around a latitude circle. Using the hydrostatic relationship, $\rho\, dz = -dp/g$, whence

$$\begin{aligned} \mathcal{H} &= -\frac{c_p}{g} \int_{x_1}^{x_2} \int_{p_s}^{0} vT\, dx\, dp \\ &= \frac{c_p}{g} \int_{x_1}^{x_2} \int_0^{p_s} vT\, dx\, dp\ , \end{aligned}$$

where p_s is surface pressure.

(c) *The figure above (note units of m K s^{-1}) shows the contribution of eddies to the atmospheric heat flux. What is actually shown is the contribution of eddies to the quantity $\overline{vT}$, which in the above notation is*

$$\overline{vT} = \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} vT\, dx$$

(i.e., the zonal average of vT), where $x_2 - x_1 = 2\pi a \cos\varphi$, where a is the Earth radius and φ latitude. Use this figure to estimate the net northward heat flux by eddies across 45°N. Compare this with the requirement (from the Earth's radiation budget) that the net (atmosphere and ocean) heat transport must be about 5×10^{15} W.

From the given figure, I estimate that the vertically averaged contribution of eddies to $\overline{vT}$, $\left[\overline{vT}\right]$ (say), is 8Kms^{-1}. From the previ-

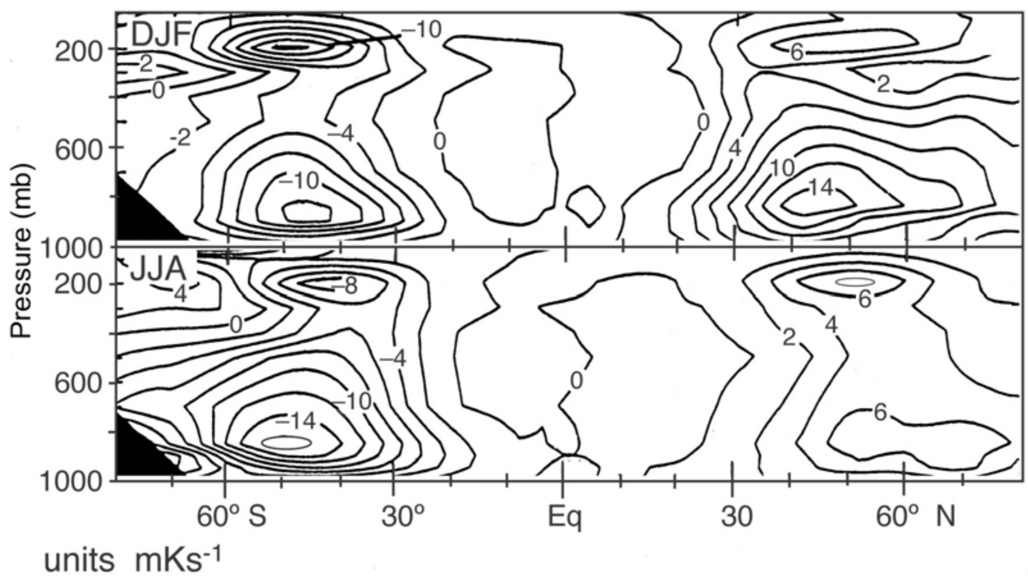


Figure 4:

ous answer, with $\Delta x = x_2 - x_1 = 2\pi a \cos\varphi = 28300$km,

$$
\begin{aligned}
\mathcal{H} &= \frac{c_p}{g}\int_{x_1}^{x_2}\int_0^{p_s} vT\;dx\;dp \\
&= \frac{c_p \Delta x\; p_s}{g}\left[\overline{vT}\right] \\
&= \frac{1004 \times 2.83 \times 10^7 \times 10^5}{9.81} \times 8\ \mathrm{W} \\
&= 2.3 \times 10^{15}\mathrm{W}.
\end{aligned}
$$

Therefore atmospheric eddies contribute about one-half of the required flux. (Note that as this latitude is north of where the Hadley cell terminates, eddies contribute essentially all of the atmospheric contribution at this latitude.)