Atmosphere, Ocean and Climate Dynamics
Answers to Chapter 7

1. *Define a streamfunction ψ for non-divergent, two-dimensional flow in a vertical plane:*
$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$
and interpret it physically.

Show that the instantaneous particle paths (streamlines) are defined by $\psi=$ const, and hence in steady flow the contours $\psi=$ const are particle trajectories. When are trajectories and streamlines not coincident?

A function $\psi=\psi\left(x,y,t\right)$ can be defined such that $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$ is <u>always</u> true. Thus if $u=-\frac{\partial \psi}{\partial y}$; $v=\frac{\partial \psi}{\partial x}$ then
$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=\frac{\partial^2\psi}{\partial y\partial x}-\frac{\partial^2\psi}{\partial y\partial x}=0.$$

ψ is called the streamfunction and is a useful way of describing the flow. Streamlines are everywhere tangential to the local flow at a given instant — they are what one would see as lines if the fluid were everywhere filled with tiny wind vanes.

The velocity can be written in vector notation
$$\mathbf{u}=(u,v)=\widehat{\mathbf{z}}\times\nabla\psi$$
where $\widehat{\mathbf{z}}$ is the unit vector in the z direction. Immediately we get:
$$\mathbf{u}\cdot\boldsymbol{\nabla}\psi=0$$
and so $\mathbf{u}$ is parallel to $\psi=$const and $|\mathbf{u}|=|\boldsymbol{\nabla}\psi|$, the speed is equal to the rate of change of ψ in the normal direction.

If $\psi=\psi\left(x,y,t\right)$ then at any instant
$$d\psi=\frac{\partial\psi}{\partial x}dx+\frac{\partial\psi}{\partial y}dy=vdx-udy.$$

Instantaneous paths are defined by $\frac{dy}{dx} = \frac{v}{u}$ and so $d\psi = 0$. In other words $\psi =$ const along a streamline.

If the flow is steady then streamlines $\equiv$ trajectories (the paths followed by individual particles of fluid): thus ψ =const define trajectories in steady flow.

If the flow is not steady then streamlines are not trajectories.

2. *What is the pressure gradient required to maintain a geostrophic wind at a speed of $v = 10\,\mathrm{m\,s^{-1}}$ at $45°N$? In the absence of a pressure gradient show that air parcels flow around circles in an anticyclonic sense of radius $\frac{v}{f}$.*

   We require a pressure gradient of magnitude:$\frac{\partial p}{\partial x} = \rho f v = 1\,\mathrm{kg\,m^{-3}} \times 2 \times 7.27 \times 10^{-5}\,\mathrm{s^{-1}} \times \sin(45°) \times 10\,\mathrm{m\,s^{-1}} = 1.028\,1 \times 10^{-3}\,\mathrm{kg\,m^{-2}\,s^{-2}}$ which is equivalent to a Δp of 10hPa in 1000 km.

   To demonstrate inertial circles, see Section 6.6.4, page 98 and note that $2\Omega \longrightarrow f$.

3. *Draw schematic diagrams showing the flow, and the corresponding balance of forces, around centers of low and high pressure in the midlatitude southern hemisphere. Do this*

   (a) *for the geostrophic flow (neglecting friction)*
   Southern hemisphere ($f < 0$). Under geostrophic balance, forces are as shown in Fig.1: Flow is clockwise (cyclonic) around a low pressure center; anticlockwise (anticyclonic) around a high.

   (b) *for the subgeostrophic flow in the near-surface boundary layer.*
   You should modify Fig.7.24 in the case $f < 0$ and remember that subgeostrophic flow is directed from high to low pressure.

4. *Consider a low pressure system centered on $45°S$, whose sea level pressure field is described by*

$$p = 1000\text{hPa} - \Delta p\, e^{-r^2/R^2} ,$$

   where r is the radial distance from the center. Determine the structure of the geostrophic wind around this system; find the maximum geostrophic wind, and the radius of the maximum wind, if $\Delta p = 20hPa$,

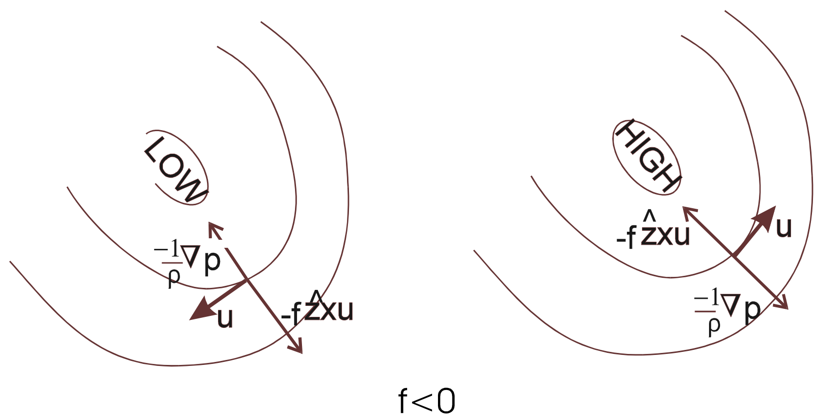


Figure 1:

$R = 500km$, and the density at sea level of interest is $1.3kg\ m^{-3}$. [Assume constant Coriolis parameter, appropriate to latitude $45°S$, across the system.]

For $p(r)$ as given, the azimuthal (anticlockwise being positive) geostrophic flow velocity is $u(r)$, where

$$\begin{aligned} u &= \frac{1}{f\rho}\frac{\partial p}{\partial r} \\ &= \frac{2r}{R^2}\frac{\Delta p}{f\rho}e^{-r^2/R^2}\ . \end{aligned}$$

This is clockwise ($u < 0$, since $f < 0$), zero at $R = 0$ and $u \to 0$ as $R \to \infty$. The function xe^{-x^2} has its maximum where $1 - 2x^2 = 0$, i.e., $x = 1/\sqrt{2}$. So the maximum velocity is found at radius $r = R/\sqrt{2} = 500/\sqrt{2} = 354$km, and its value there is

$$\begin{aligned} |u_{\max}| &= \frac{\sqrt{2}}{R}\frac{\Delta p}{|f|\,\rho}e^{-\frac{1}{2}} \\ &= \sqrt{\frac{2}{e}}\frac{2\times 10^3}{5\times 10^5\times 1.03\times 10^{-4}\times 1.3}\ \mathrm{ms}^{-1} \\ &= 25.6\ \mathrm{ms}^{-1}\ . \end{aligned}$$

5. *Write down an equation for the balance of radial forces on a parcel of fluid moving along a horizontal circular path of radius r at constant*

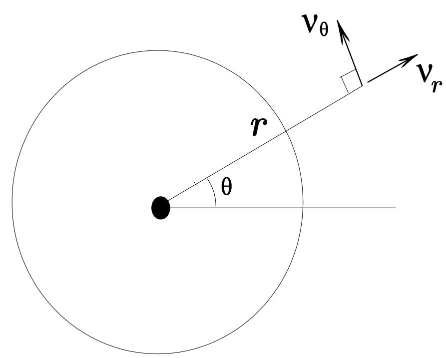


Figure 2: The velocity of a fluid parcel viewed in the rotating frame of reference: $v_{rot} = (v_\theta, v_r)$.

speed v_θ (taken positive if the flow is in the same sense of rotation as the earth).

Solve for v_θ as a function of r and the radial pressure gradient and hence show that:

(a) *if $v_\theta > 0$, the wind speed is less than its geostrophic value,*

(b) *if $|v_\theta| << fr$ then the flow approaches its geostrophic value and*

(c) *there is a limiting pressure gradient for the balanced motion when $v_\theta > -\frac{1}{2}fr$.*

Comment on the asymmetry between clockwise and anticlockwise vortices.

There is a 3-way balance of forces in the radial direction between centrifugal, Coriolis and pressure gradient forces:

$$\frac{v_\theta^2}{r} + f = g\frac{\partial h}{\partial r}$$

where h is the height of a pressure surface. This can be re-arranged to give:

$$v_\theta = \frac{g}{\left(f + \frac{v_\theta}{r}\right)}\frac{\partial h}{\partial r}$$

a. Thus the observed wind will be less than the geostrophic wind in a cyclonic situation ($v_\theta > 0$).

b. If $|v_\theta| << fr$ then $v_\theta \longrightarrow \frac{g}{f}\frac{\partial h}{\partial r}$, the geostrophic value.

c. Solving for v_θ we find that:

$$v_\theta = -\frac{1}{2}fr + \left(\frac{1}{4}f^2r^2 + gr\frac{\partial h}{\partial r}\right)^{\frac{1}{2}}$$

where the positive root has been chosen. If $v_\theta > -\frac{1}{2}fr$ then $\left(\frac{1}{4}f^2r^2 + gr\frac{\partial h}{\partial r}\right)^{\frac{1}{2}} > 0$ and so $\frac{\partial h}{\partial r} > -\frac{1}{4g}f^2r$ placing a limit on the pressure gradient. Thus there is no limit on the intensity of cyclones, but there is a limit to how intense anticyclones can get.

6. *(i) A typical hurricane at, say, $30°$ latitude may have low-level winds of $50ms^{-1}$ at a radius of $50km$ from its center: do you expect this flow to be geostrophic?*

At 30^oN, the Coriolis parameter is $f = 2\Omega \sin 30^o = \Omega = 7.27 \times 10^{-5}\text{s}^{-1}$, so the Rossby number for a hurricane with winds 50ms^{-1} at a radius of 50km is

$$R = \frac{50}{7.27 \times 10^{-5} \times 5 \times 10^4} \simeq 13.8 \ .$$

This number is not small, so the flow is not expected to be geostrophic.

(ii) Two weather stations near $45°N$ are $400km$ apart, one exactly to the northeast of the other. At both locations, the $500hPa$ wind is exactly southerly at $30ms^{-1}$. At the north-eastern station, the height of the $500hPa$ surface is $5510m$; what is the height of this surface at the other station?

Assuming geostrophic balance (and using our standard notation, in pressure coordinates)

$$(u, v) = \frac{g}{f}\left(-\frac{\partial z}{\partial y}, \frac{\partial z}{\partial x}\right) \ .$$

Given that the flow is 30ms^{-1} southerly, the 500hPa height gradient is

$$\begin{aligned}\left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) &= \left(\frac{fv}{g}, -\frac{fu}{g}\right) \\ &= \left(\frac{1.03 \times 10^{-4} \times 30}{9.81}, 0\right) = \left(3.15 \times 10^{-4}, 0\right) \ \text{[dimensionless]}.\end{aligned}$$

If z_e and z_w are the 500hPa heights at the eastern and western stations, respectively, then, assuming the components of the vector separating the two stations, $\delta x = x_e - x_w = 400/\sqrt{2}$m and $\delta y = y_e - y_w = 400/\sqrt{2}$m, are small enough,

$$\begin{aligned}\delta z &= z_e - z_w = \frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y = \frac{\partial z}{\partial x}\delta x \\ &= 3.15 \times 10^{-4} \times \frac{4 \times 10^5}{\sqrt{2}} = 89\text{m}.\end{aligned}$$

Therefore the height at the western station is $5510 - 89 = 5421$m.

7. *Write down an expression for the centrifugal acceleration of a ring of air moving uniformly along a line of latitude with speed u relative to the earth, which itself is rotating with angular speed Ω. Interpret the terms in the expression physically. By hypothesizing that the relative centrifugal acceleration resolved parallel to the earth's surface is balanced by a meridional pressure gradient, deduce the geostrophic relationship*

$$fu + \frac{1}{\rho}\frac{\partial p}{\partial y} = 0$$

(in our usual notation and where $dy = a d\varphi$).

Consider the ring of air moving eastward at speed u relative to the underlying rotating earth shown in the figure.There is a centrifugal acceleration directed outwards perpendicular to the earth's axis of rotation, the vector A:

$$\frac{V^2}{r} = \frac{(u + \Omega r)^2}{r} = \Omega^2 r + 2\Omega u + \frac{u^2}{r} \tag{1}$$

Here V is the 'absolute' velocity the fluid has viewed from an observer fixed in space looking back at the earth. Let's now consider the terms in turn:

- $\Omega^2 r$ — this is the centrifugal acceleration acting on a particle fixed to the earth. As discussed above, this acceleration is included in the gravity which is usually measured and is the reason that the earth is not a perfect sphere.

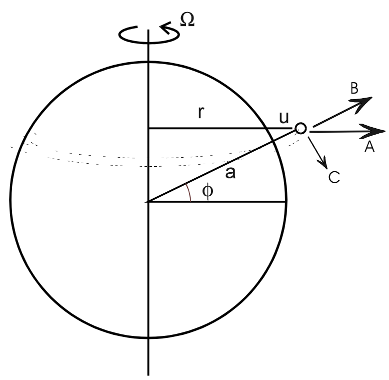


Figure 3:

- $2\Omega u + \frac{u^2}{r}$ - the additional centrifugal acceleration due to motion relative to the earth. Note that if $\frac{u}{\Omega r} << 1$, we may neglect the term in u^2. For the earth $R_o = \frac{u}{\Omega r} \sim 0.02$ and so the $2\Omega u$ term dominates. It is directed outward perpendicular to the axis of rotation and can be resolved: perpendicular to the earth's surface - vector B in the diagram — and parallel to the earth's surface — vector C in the diagram.

Component B changes the weight of the ring slightly — it is very small compared to g, the acceleration due to gravity, and so unimportant.

Component C, parallel to the earth's surface, is the Coriolis acceleration:

$$2\Omega \sin\varphi \times u$$

So there is a centrifugal force directed toward the equator because of the motion of the ring of air relative to the earth. It is this force that balances the pressure gradient force associated with the sloping isobaric surfaces induced by the pole-equator temperature gradient.

Let's postulate a balance between the Coriolis force and the pressure-gradient force directed from equator to pole associated with the tilted isobaric surfaces — see Fig.4.

$$\underbrace{\rho a d\varphi dz}_{mass} \times \underbrace{2\Omega \sin\varphi u}_{acceleration} = \underbrace{-\frac{\partial p}{\partial \varphi} d\varphi dz}_{p\_grad}$$

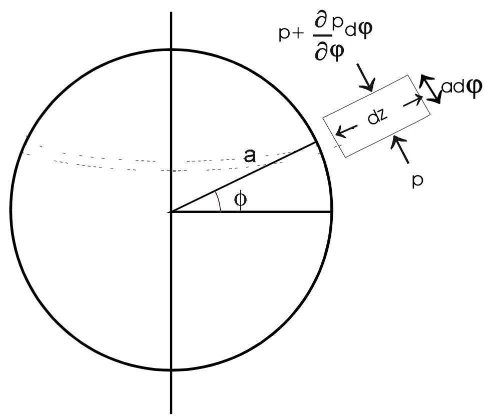


Figure 4:

Introducing a coordinate y which points northwards on the earth's surface, $dy = ad\varphi$, the above reduces to:

$$fu + \frac{1}{\rho}\frac{\partial p}{\partial y} = 0$$

where $f = 2\Omega \sin\varphi$ is the Coriolis parameter, which is the answer we seek.

8. *The vertical average (with respect to log pressure) of atmospheric temperature below the* $200hPa$ *pressure surface is about* $265K$ *at the equator and* $235K$ *at the winter pole. Calculate the equator-to-winter-pole height difference on the* $200hPa$ *pressure surface, assuming surface pressure is* $1000hPa$ *everywhere. Assuming that this pressure surface slopes uniformly between* 30^o *and* 60^o *latitude and is flat elsewhere, use the geostrophic wind relationship (zonal component) in pressure coordinates,*

$$u = -\frac{g}{f}\frac{\partial z}{\partial y}\ .$$

to calculate the mean eastward geostrophic wind on the $200hPa$ *surface at* 45^o *latitude in the winter hemisphere. Here* $f = 2\Omega\sin(lat)$ *is the Coriolis parameter,* g *is the acceleration due to gravity,* z *is the height* *of a pressure surface and $dy = a \times d(lat)$ where a is the radius of the earth is a northward pointing coordinate.*

From hydrostatic balance

$$\frac{\partial p}{\partial z} = -g\rho \ ,$$

we get, using the gas law,

$$dz = -\frac{dp}{g\rho} = -\frac{RT}{g}\frac{dp}{p} \ .$$

Integrating from the surface (where $z = 0$ and $p = 1000$hPa) to 200hPa,

$$\begin{aligned} \int_0^{z_{200}} dz &= z_{200} = -\frac{R}{g}\int_{1000}^{200} T\frac{dp}{p} \\ &= \frac{R}{g}\int_{p=200}^{p=1000} T \ d\ln p \\ &= \frac{R}{g}\langle T \rangle \ln\left(\frac{1000}{200}\right) \ , \end{aligned}$$

where $\langle T \rangle$ is the mean temperature with respect to log pressure. Given that $\langle T \rangle = 265$K at the equator and 235K at the winter pole, the difference in mean temperatures is $\Delta \langle T \rangle = 30$K and the height difference between pole and equator is therefore

$$\begin{aligned} \Delta z_{200} &= \frac{R}{g}\Delta\langle T \rangle \ln\left(\frac{1000}{200}\right) \\ &= \frac{287}{9.81} \times 30 \times \ln 5 \\ &= 1413\text{m}, \end{aligned}$$

with the equator being high and the pole low. If this height difference is concentrated uniformly between 30^o and 60^o latitude, a distance of $\pi a/6$, where a is the Earth radius, the height gradient there is

$$\frac{\partial z}{\partial y} = -1413 \times \frac{6}{\pi \times 6.37 \times 10^6} = -4.2365 \times 10^{-4} \ .$$

Using the geostrophic wind relationship (zonal component) in pressure coordinates,

$$u = -\frac{g}{f}\frac{\partial z}{\partial y} \ .$$

At 45^o, where $f = 2\Omega \sin 45^o = \sqrt{2} \times 7.27 \times 10^{-5} = 1.03 \times 10^{-4}\text{s}^{-1}$, the wind is

$$\begin{aligned} u &= \frac{9.81}{1.03 \times 10^{-4}} \times 4.2365 \times 10^{-4} \\ &= 40.35\text{ms}^{-1}. \end{aligned}$$

9. *From the pressure coordinate thermal wind relationship, and approximating*
$$\frac{\partial u}{\partial p} \simeq \frac{\partial u/\partial z}{\partial p/\partial z} ,$$
show that, in geometric height coordinates,
$$f\frac{\partial u}{\partial z} \simeq -\frac{g}{T}\frac{\partial T}{\partial y}.$$

The thermal wind relationship for the zonal flow component is, see Eq.(7.23),

$$\frac{\partial u}{\partial p} = \frac{R}{fp}\frac{\partial T}{\partial y} .$$

Using the hydrostatic relationship,

$$\frac{\partial u}{\partial p} \simeq \frac{\partial u/\partial z}{\partial p/\partial z} = -\frac{1}{g\rho}\frac{\partial u}{\partial z} .$$

whence

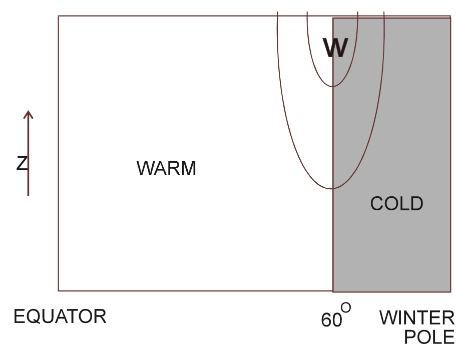
$$\frac{\partial u}{\partial z} \simeq -\frac{Rg\rho}{fp}\frac{\partial T}{\partial y} .$$

But, from the perfect gas law, $R\rho/p = 1/T$, so

$$f\frac{\partial u}{\partial z} \simeq -\frac{g}{T}\frac{\partial T}{\partial y} .$$

The winter polar stratosphere is dominated by the "polar vortex," a strong westerly circulation at about 60^o *latitude around the cold pole, as depicted schematically in the figure. (This circulation is the subject of considerable interest, as it is within the polar vortices—especially that over Antarctica in southern winter and spring—that most ozone depletion is taking place.) Assuming that the temperature at the pole is (at all heights)* $50K$ *colder at* 80^o

latitude than at 40^o latitude (and that it varies uniformly in between), and that the westerly wind speed at $100hPa$ pressure and 60^o latitude is $10ms^{-1}$, use the thermal wind relation to estimate the wind speed at $1hPa$ pressure and 60^o latitude.



Assuming geostrophic flow, thermal wind balance gives

$$\frac{\partial \mathbf{u}}{\partial p} = -\frac{R}{fp}\widehat{\mathbf{z}} \times \nabla T \ .$$

For uniform temperature gradient

$$\frac{\partial T}{\partial y} = -\frac{50}{4.44 \times 10^6} = -1.1 \times 10^{-5}\mathrm{Km}^{-1}$$

we have

$$\frac{\delta u}{\delta \ln p} = -\frac{R}{f}\frac{\partial T}{\partial y}.$$

At 60^o latitude, $f = 2\Omega \sin 60^o = 1.26 \times 10^{-4}\mathrm{s}^{-1}$, whence

$$u_1 - u_{100} = \frac{287}{1.26 \times 10^{-4}} \times 1.1 \times 10^{-5} \times \ln 100 = 115\mathrm{ms}^{-1}.$$

Given that $u_{100} = 10\mathrm{ms}^{-1}$, we get $u_1 = 125\mathrm{ms}^{-1}$.

10. *Starting from Eq.(7.24), show that the thermal wind equation can be written in terms of potential temperature, Eq.(4.17), thus:*

$$\left(\frac{\partial u_g}{\partial p}, \frac{\partial v_g}{\partial p}\right) = \frac{1}{\rho f \theta}\left(\left(\frac{\partial \theta}{\partial y}\right)_p, -\left(\frac{\partial \theta}{\partial x}\right)_p\right).$$

Taking the x component of Eq.(7.24) we have, using Eq.(4.17):

$$\begin{aligned}\frac{\partial u_g}{\partial p} &= \frac{R}{fp}\left(\frac{\partial T}{\partial y}\right)_p = \frac{R}{fp}\left(\frac{\partial}{\partial y}\left[\theta\left(\frac{p}{p_o}\right)^\kappa\right]\right)_p = \frac{R}{fp}\left(\frac{p}{p_o}\right)^\kappa\left(\frac{\partial \theta}{\partial y}\right)_p \\ &= \frac{1}{f\rho T}\left(\frac{p}{p_o}\right)^\kappa\left(\frac{\partial \theta}{\partial y}\right)_p = \frac{1}{\rho f\theta}\left(\frac{\partial \theta}{\partial y}\right)_p\end{aligned}$$

where we have also used the ideal gas law.

11. *Fig.7.30 shows, schematically, the surface pressure contours (solid) and mean 1000hPa-500hPa temperature contours (dashed), in the vicinity of a typical northern hemisphere depression (storm). "L" indicates the low pressure center. Sketch the directions of the wind near the surface, and on the 500hPa pressure surface. (Assume that the wind at 500hPa is significantly larger than at the surface.) If the movement of the whole system is controlled by the 500hPa wind (i.e., it simply gets blown downstream by the 500hPa wind), how do you expect the storm to move? [Use density of air at 1000hPa = $1.2kg\ m^{-3}$; rotation rate of Earth = $7.27 \times 10^{-5} s^{-1}$; gas constant for air = $287 J\ kg^{-1}$]*

    Note that the contour interval in Fig.7.30 was, by mistake, not indicated but is 2hPa for the surface pressure and 2 °C for temperature.

    Assuming geostrophic flow just above the surface boundary layer,

$$\mathbf{u} = \frac{1}{f\rho}\widehat{\mathbf{z}} \times \nabla p$$

    At 45^{o}N, $f = 2\Omega \sin 45^o = 1.0 \times 10^{-4}\mathrm{s}^{-1}$. At point C (storm center), $\nabla p = 0$, so $|\mathbf{u}_{1000}|_C = 0$ there. At point A, distance between 2hPa isobars is about 70km, so

$$|\mathbf{u}_{1000}|_A = \frac{1}{(1.2) \times (1.0 \times 10^{-4})}\frac{200}{7 \times 10^4} = 24\mathrm{ms}^{-1}\ .$$

    At point B, separation of isobars is about 40km, so

$$|\mathbf{u}_{1000}|_B = \frac{1}{(1.2) \times (1.0 \times 10^{-4})}\frac{200}{4 \times 10^4} = 42ms^{-1}\ .$$

    The winds are cyclonic, and along isobars, as shown by the thin solid arrows.

From the thermal wind equation,

$$\frac{\partial \mathbf{u}}{\partial p} = -\frac{R}{fp}\widehat{\mathbf{z}} \times \nabla T$$

we have

$$\begin{aligned} \mathbf{u}_{500} - \mathbf{u}_{1000} &= -\frac{R}{f}\int \widehat{\mathbf{z}} \times \nabla T \, d\ln p \\ &= \frac{R}{f}\widehat{\mathbf{z}} \times \nabla \langle T \rangle \, \ln 2 \end{aligned}$$

where $\langle T \rangle$ is the 1000-500hPa mean temperature. Near point B, the separation between T contours is about 50km, whence

$$|\mathbf{u}_{500} - \mathbf{u}_{1000}|_B = \frac{287}{1 \times 10^{-4}} \frac{2}{5 \times 10^4} \ln 2 = 80\text{ms}^{-1} \ .$$

Since I have chosen point A to be where the isobars/height contours are parallel, the low-level and thermal winds are antiparallel, so

$$|\mathbf{u}_{500}|_B = 38\text{ms}^{-1} \ .$$

The wind vectors at 500hPa are shown by the heavy arrows. Near point A, the separation is about 60km, and

$$|\mathbf{u}_{500} - \mathbf{u}_{1000}|_A = \frac{287}{1 \times 10^{-4}} \frac{2}{6 \times 10^4} \ln 2 \ = 66\text{ms}^{-1} \ .$$

The 1000hPa flow and thermal wind are parallel, so

$$|\mathbf{u}_{500}|_A = 90\text{ms}^{-1} \ .$$

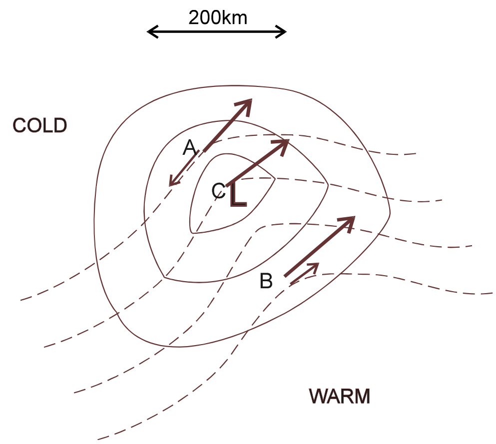
At point C, the T contour separation is also about 60km, so

$$|\mathbf{u}_{500} - \mathbf{u}_{1000}|_C = 66\text{ms}^{-1} \ .$$

Since the low-level flow is zero,

$$|\mathbf{u}_{500}|_C = 66\text{ms}^{-1} \ .$$

The storm center therefore should move approximately northeastwards at about 66ms^{-1}.


200km
COLD
A
C
L
B
WARM