Atmosphere, Ocean and Climate Dynamics
Answers to Chapter 4

1. *Show that the buoyancy frequency, Eq.(4.22), may be written in terms of the environmental temperature profile thus*

$$N^2 = \frac{g}{T_E}\left(\frac{dT_E}{dz} + \Gamma_d\right)$$

where Γ_d is the dry adiabatic lapse rate.

Using the definition (4.17) for the environmental potential temperature θ_E, Eq. (4.22) gives

$$\begin{aligned}
N^2 &= \frac{g}{\theta_E}\frac{d\theta_E}{dz} \\
&= \frac{g}{T_E}\left(\frac{p_E}{p_0}\right)^{\kappa}\left[\frac{dT_E}{dz}\left(\frac{p_0}{p_E}\right)^{\kappa} - \kappa p_0\left(\frac{p_0}{p_E}\right)^{\kappa-1} T_E\frac{dp_E}{dz}\right] \\
&= \frac{g}{T_E}\left[\frac{dT_E}{dz} - \kappa\frac{T_E}{p_E}\frac{dp_E}{dz}\right] .
\end{aligned}$$

But, from the ideal gas law (1.1), $T_E/p_E = (R\rho_E)^{-1}$, while hydrostatic balance (3.3) we have $dp_E/dz = -g\rho_E$, whence

$$N^2 = \frac{g}{T_E}\left[\frac{dT_E}{dz} + \frac{g\kappa}{R}\right] = \frac{g}{T_E}\left[\frac{dT_E}{dz} + \Gamma_d\right] ,$$

where $\Gamma_d = g\kappa/R = g/c_p$ is the adiabatic lapse rate.

2. *From the temperature (T) profile shown in Fig.4.9:*

   (a) *estimate the tropospheric lapse rate and compare to the dry adiabatic lapse rate.*

   The mean temperature gradient, with respect to pressure, over the layer $700 - 300$h Pa is approximately $\partial T/\partial p \simeq 45$ K/400 h Pa $\simeq 1.1 \times 10^{-3}$ K Pa^{-1}. Using the ideal gas law (1.1), a mean temperature (see part (b) of this question) of 260 K and a mean pressure $p_0 = 500$ h Pa, the mean density of the layer is $\rho_o =$

$p_o/(RT_o) = 5 \times 10^4/(387 \times 260) = 0.50$ kg m^{-3}. Hence, using hydrostatic balance (3,3),

$$\frac{\partial T}{\partial z} = \frac{\partial T}{\partial p}\frac{\partial p}{\partial z} = -g\rho\frac{\partial T}{\partial p} \simeq -9.81 \times 0.50 \times 1.1 \times 10^{-3} \simeq -5.4 \times 10^{-3} \text{ K m}^{-1},$$

i.e., the mean lapse rate is about 5.4 K km^{-1}, compared with the adiabatic lapse rate $\Gamma_d = g/c_p = 9.8$ K m^{-1}. So this temperature profile is stable to unsaturated convection.

(b) *estimate the pressure scale height RT_0/g, where T_o is the mean temperature over the* 700 mbar *to* 300 mbar *layer.*

Estimating the mean temperature in the layer to be $T_o \simeq 260$ K, the corresponding scale height is $H = RT_o/g = 287 \times 260/9.81 = 7.6$ km.

(c) *estimate the period of buoyancy oscillations in mid-troposphere.*

Using the result from Q.1, the buoyancy frequency N is given by

$$N^2 = \frac{g}{T_E}\left[\frac{dT_E}{dz} + \Gamma_d\right].$$

Substituting from our estimates,

$$N^2 \simeq \frac{9.81}{260}[-5.4 + 9.8] \times 10^{-3} = 1.7 \times 10^{-4} \text{ s}^{-2},$$

whence the buoyancy period is $2\pi/N \simeq 2\pi/\sqrt{1.7 \times 10^{-4}} \simeq 480$ s.

3. *Consider the laboratory convection experiment described in section 4.2.4. The thermodynamic equation (horizontally averaged over the tank) can be written:*

$$\rho c_p \frac{dT}{dt} = \frac{\mathcal{H}}{h} \qquad (4.31)$$

where h is the depth of the convection layer — see Fig.4.28 — H is the (constant) heat flux coming in at the bottom from the heating pad, ρ is the density, c_p is the specific heat, t is time and T is temperature.

We observe that the temperature in the convection layer is almost homogeneous and 'joins on' to the linear stratification into which the convection is burrowing, as sketched in Fig.4.28. Show that if this is the case, Eq.(4.31) can be written thus:

$$\frac{\rho c_p \overline{T_z}}{2}\frac{d}{dt}h^2 = \mathcal{H}$$

where $\overline{T_z}$ is assumed to be constant.

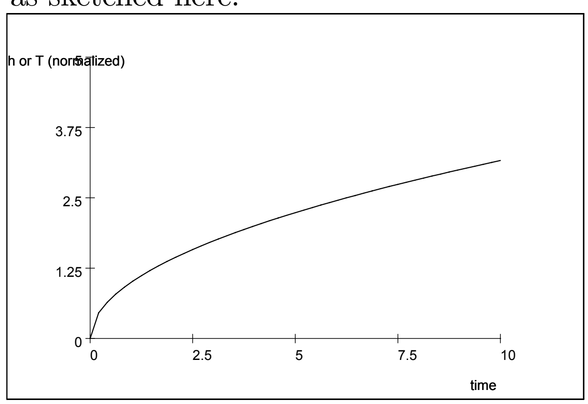
If the stratification is linear ($\overline{T_z}$ = constant) and the temperature of the boundary layer joins on to it (as sketched in the figure) then the temperature of the boundary layer is directly related to its depth h and, to within a constant, can be written, $T = \overline{T_z}h$. Thus, Eq.(1) can be rearranged to yield the equation above.

Solve the above equation for $h(t)$ and hence show that the depth and temperature of the convecting layer increases like $\sqrt{t}$. Sketch the form the solution for $h(t)$ and $T(t)$.

Solving we find that:

$$h^2 = \frac{2\mathcal{H}}{\rho c_p \overline{T_z}} t \text{ or } h = \sqrt{\frac{2\mathcal{H}}{\rho c_p \overline{T_z}}} \sqrt{t}$$

if $\mathcal{H}$ and $\overline{T_z}$ are constant. Thus h and, in view of $T = \overline{T_z}h$, T both increase like $\sqrt{t}$, as sketched here.



(a) *Is your solution consistent with the plot of the temperature evolution from the laboratory experiment shown in Fig.4.8?*

Yes, the experimental data is consistent with a $\sqrt{t}$ dependence.

(b) *How would T have varied with time if initially the water in the tank had been of uniform temperature (i.e. was unstratified)? You may assume that the water remains well mixed at all times and so is of uniform temperature.*

In this case the depth of the fluid undergoing convective mixing is constant and Eq.(4.31) therefore implies that T increases linearly with time.

4. *Consider an atmospheric temperature profile at dawn with a temperature discontinuity (inversion) at 1km, and a tropopause at 11km, such that*

$$T(z) = \begin{cases} 10^o\mathrm{C}, & z < 1\mathrm{km} \\ [15 - 8\,(z-1)]^o\,\mathrm{C} & 1 < z < 11\mathrm{km} \\ -65^o\mathrm{C} & z > 11\mathrm{km} \end{cases}$$

(where here z is expressed in km). Following sunrise at 6 a.m. until 1 p.m., the surface temperature steadily increases from its initial value of $10^o C$ at a rate of $3^o C$ per hour. Assuming that convection penetrates to the level at which air parcels originating at the surface attain neutral buoyancy, describe the evolution during this time of convection

(a) *if the surface air is completely dry;*

(b) *if the surface air is saturated.*

[You may assume that unsaturated (saturated) air parcels follow the dry (moist) adiabatic lapse rate of $10\,\mathrm{K\,km^{-1}}$ ($7\,\mathrm{K\ \ km^{-1}}$) under all conditions (even at finite displacements).]

a) As the ground starts to warm after dawn (temperature $T_s = (10+3t)^\circ\mathrm{C}$, where t is time in hours after 6am), parcels from the surface will be positively buoyant until they reach an altitude z_{top}(km) such that

$$T_s - 10z_{top} = T_{init}.$$

z_{top} will reach 1km when $T_s = 20^\circ\mathrm{C}$, at 9:20am. Until that time, convection cannot extend above a level z_{top}, which is less than 1km. (E.g., at 8:00am, $T_s = 16^\circ\mathrm{C}$, $z_{top} = 0.6$km.) z_{top} remains at 1km until $T_s = 25^\circ\mathrm{C}$, at 11:00am. After that time, convection will extend beyond 1km, in fact to z_{top}, where

$$T_s - 10z_{top} = 23 - 8z_{top}\ ,$$

i.e.,

$$z_{top} = \frac{1}{2}\,(T_s - 23)\ .$$

Thus, the depth of convection will gradually increase as T_s increases (since the background lapse rate above 1km, -8K km^{-1}, is greater than $-\Gamma_d$), reaching 4km at 1:00pm, when $T_s = 31°\text{C}$.

b) The maximum height of convection, z_{top} (the level of neutral buoyancy), is now defined by

$$T_s - 7z_{top} = T_{init} .$$

Qualitatively, the early development of convection is similar to that of the dry case. Convection first reaches 1km when $T_s = 17°\text{C}$, at 8:20am, and remains at that level until $T_s = 22°\text{C}$, at 10:00am. After 10:00, however, the situation changes rapidly. If $T_s > 22°\text{C}$ then, after the parcel passes 1km altitude, it remains positively buoyant all the way to the tropopause, because the environmental lapse rate above 1km is conditionally unstable ($dT/dz < -\Gamma_s$). In fact, after 10:00, the level of neutral buoyancy is given by

$$T_s - 7z_{top} = -65 .$$

Just after 10:00 ($T_s = 22°\text{C}$), therefore, convection extends all the way to $12\frac{3}{7}$km. The reason for the dramatic difference from the unsaturated case is that the lapse rate above the inversion exceeds the saturated lapse rate: the profile is conditionally unstable. Thus, once surface air becomes warm enough to penetrate the inversion, it rises all the way to the tropopause. After this time, z_{top} increases slowly to a value $13\frac{5}{7}$km at 1:00pm, when $T_s = 31°\text{C}$.

5. *For a perfect gas undergoing changes dT in temperature and dV in specific volume, the change ds in specific entropy, s, is given by*

$$T\,ds = c_v dT + p\,dV .$$

(a) *Hence, for unsaturated air, show that potential temperature θ*

$$\theta = T\left(\frac{p_0}{p}\right)^{\kappa} ,$$

(notation defined in notes) is a measure of specific entropy; specifically, that

$$s = c_p \ln\theta + \text{constant} .$$

where c_v and c_p are specific heats at constant volume and constant pressure, respectively.
Change in entropy, ds, given by

$$T\,ds = c_v\,dT + p\,dV$$

where specific volume change $dV = d(1/\rho) = d(RT/p) = \frac{R}{p}dT - \frac{1}{p\rho}dp$. So

$$p\,dV = R\,dT - \frac{1}{\rho}dp\ .$$

Hence

$$T\,ds = c_p dT - \frac{1}{\rho}dp\ ,$$

where $c_p = c_v + R$, and so

$$ds = c_p\frac{dT}{T} - R\frac{dp}{p}\ . \qquad (1)$$

Now, potential temperature is

$$\theta = T\left(\frac{p_0}{p}\right)^{\kappa},$$

where p_0 is a constant. Therefore

$$d\theta = dT\left(\frac{p_0}{p}\right)^{\kappa} - \frac{\kappa T}{p}\left(\frac{p_0}{p}\right)^{\kappa} dp$$

or

$$\frac{d\theta}{\theta} = d\ln\theta = \frac{dT}{T} - \kappa\frac{dp}{p}\ .$$

So

$$c_p d\ln\theta = c_p\frac{dT}{T} - R\frac{dp}{p}\ , \qquad (2)$$

since $\kappa = R/c_p$. Comparing (1) and (2),

$$ds = c_p d\ln\theta\ ,$$

or $s = c_p\ln\theta$+constant .

(b) *Show that if the environmental lapse rate is dry adiabatic, then it has constant potential temperature.*

From part (a) we have $s = c_p \ln \theta +$ constant, thus $ds = c_p d(\ln \theta) = c_p \frac{d\theta}{\theta}$. When the parcel is moved adiabatically, no heat is exchanged with the surroundings - $dQ = Tds = 0$ - its entropy is conserved. From the relation $ds = c_p d(\ln \theta) = c_p \frac{d\theta}{\theta}$,we see that $d\theta = 0$, i.e. the potential temperature is conserved when the parcel is moved adiabatically.

6. *Investigate under what conditions we may approximate* $\frac{L}{c_p T} dq_*$ *by* $d\left(\frac{Lq_*}{c_p T}\right)$ *in the derivation of Eq.(4.30). Is this a good approximation in typical atmospheric conditions?*

The key assumption made in the derivation of (4.30) is that, in the expression

$$d\left(\frac{Lq}{c_p T}\right) = \frac{L}{c_p T} dq - \frac{Lq}{c_p T^2} dT \ ,$$

the second term may be ignored compared to the first. If we rewrite the equation as

$$d\left(\frac{Lq}{c_p T}\right) = \frac{Lq}{c_p T}\left(\frac{dq}{q} - \frac{dT}{T}\right) \ ,$$

we can see that the neglect of the second term is tantamount to assuming that $|\delta (\ln T)| \ll |\delta (\ln q)|$. Now, from the global mean moisture profile shown in Fig. 3.3, over the altitude range from the surface to 400 hPa q decreases from about 17 to about $1\,\mathrm{g\ kg^{-1}}$, whence $|\delta (\ln q)| \simeq \ln (0.017) - \ln (0.001) \simeq 2.8$, whereas, over the same height range, T varies from about 288 to about 253 K, whence $|\delta (\ln T)| \simeq \ln (288) - \ln (253) \simeq 0.13$. Hence, for this example, neglect of the second term results in a fractional error of a little less than 5%.

7. *Assume the atmosphere is isothermal with temperature* $280K$. *Determine the potential temperature at altitudes of* $5\,\mathrm{km}$, $10\,\mathrm{km}$, *and* $20\,\mathrm{km}$ *above the surface. If an air parcel were moved adiabatically from* $10km$ *to* $5km$, *what would its temperature be on arrival?*

For an isothermal atmosphere with temperature T, pressure varies with altitude as

$$p(z) = p_s \exp\left(-\frac{z}{H}\right) \ ,$$

where p_s is surface pressure (1000hPa) and the scale height is $H = RT/g$ (in our usual notation). Given $T = 280$K,

$$H = \frac{287 \times 280}{9.81}\text{m} = 8.192\text{km} .$$

Potential temperature is

$$\theta = T\left(\frac{p_s}{p}\right)^{\kappa} ,$$

where $\kappa = 2/7$.

$\mathbf{z = 5km}$:

$$p(5\text{km}) = 1000\exp(-\frac{5}{8.192}) = 543\text{hPa}.$$

Therefore $\theta(5\text{km}) = 280 \times \left(\frac{1000}{543}\right)^{\frac{2}{7}} = 333$K.

$\mathbf{z = 10km:}$

$$p(10\text{km}) = 1000\exp(-\frac{10}{8.192}) = 295\text{hPa}.$$

$\rightarrow \theta(10\text{km}) = 280 \times \left(\frac{1000}{295}\right)^{\frac{2}{7}} = 397$K.

$\mathbf{z = 20km:}$

$$p(20\text{km}) = 1000\exp(-\frac{20}{8.192}) = 87\text{hPa}.$$

$\rightarrow$

$$\theta(20\text{km}) = 280 \times \left(\frac{1000}{87}\right)^{\frac{2}{7}} = 563\text{K}.$$

In an air parcel is moved adiabatically from 10km to 5km, it will conserve its potential temperature. Therefore, on arrival at 5km, it will have the same θ as when it left 10km, *i.e.*, 397K. Its temperature on arrival is therefore

$$T = \theta\left(\frac{p}{p_s}\right)^{\kappa} = 397 \times \left(\frac{543}{1000}\right)^{\frac{2}{7}} = 333\text{K} = 60°\text{C (!)}.$$

8. *Somewhere (in a galaxy far, far away) there is a planet whose atmosphere is just like that of the Earth in all respects but one — it contains no moisture. The planet's troposphere is maintained by convection to be neutrally stable to vertical displacements. Its stratosphere is in radiative equilibrium, at a uniform temperature* $-80°\,C$*, and temperature is continuous across the tropopause. If the surface pressure is* $1000hPa$*, and equatorial surface temperature is* $32°\,C$*, what is the pressure at the equatorial tropopause?*

   If the troposphere is neutrally stable to dry convection, then potential temperature θ is uniform with height within the troposphere. Since $\theta = T = 305\mathrm{K}$ at the surface ($p = p_0$), θ has this value at all heights. At the tropopause, the temperature is equal to the stratospheric value, $-80°\mathrm{C} = 193\mathrm{K}$, and its potential temperature must also be $\theta = 305\mathrm{K}$. Since $\theta = T\,(p/p_0)^{-\kappa}$, where $\kappa = 2/7$ for the atmosphere, the pressure at the tropopause must therefore be

$$p = p_0 \left(\frac{T}{\theta}\right)^{\frac{1}{\kappa}} = 1000 \times \left(\frac{193}{305}\right)^{\frac{7}{2}} = 201.6 \text{ hPa}.$$

9. *Compare the dry-adiabatic lapse rate on Jupiter with that of Earth given that the gravitational acceleration on Jupiter is* $26\,\mathrm{m\,s^{-2}}$ *and its atmosphere is composed almost entirely of hydrogen and therefore has a different value of* c_p*.*

   The c_p of hydrogen gas is $14 \times 10^3\,\mathrm{J\ kg^{-1}\ K^{-1}}$ and so the ratio of dry-adiabatic lapse rates on Jupiter and Earth is, using Eq.(4.14):

$$\frac{c_{p_{\mathrm{Jupiter}}}}{c_{p_{\mathrm{Earth}}}} = \frac{g_{\mathrm{Jupiter}}}{g_{\mathrm{Earth}}} \times \frac{c_{p_{\mathrm{Earth}}}}{c_{p_{\mathrm{Jupiter}}}} = \frac{26}{9.81} \times \frac{1005}{14 \times 10^3} = 0.19.$$

10. *In Section 3.3 we showed that the pressure of an isothermal atmosphere varies exponentially with height. Consider now an atmosphere with uniform potential temperature. Find how pressure varies with height, and show in particular that such an atmosphere has a discrete top (where* $p \to 0$*) at altitude* $RT_o/\,(\kappa g)$*, where* R*,* κ*, and* g *have their usual meanings, and* T_0 *is the temperature at* $1000hPa$ *pressure.*
    In hydrostatic balance, and for a perfect gas,

$$\frac{\partial p}{\partial z} = -g\rho = -g\frac{p}{RT}\ .$$

An atmosphere with uniform potential temperature has

$$\theta = T\left(\frac{p_0}{p}\right)^{\kappa} = \text{constant} = T_0 \ ,$$

where T_0 is the temperature where $p = p_0$ (1000hPa). Hence $T = T_0\,(p/p_0)^{\kappa}$, so

$$\frac{\partial p}{\partial z} = -g\frac{p_0^{\kappa}p^{1-\kappa}}{RT_0} \ .$$

Hence

$$p^{\kappa-1}dp = -\frac{gp_0^{\kappa}}{RT_0}\ dz \ ,$$

so

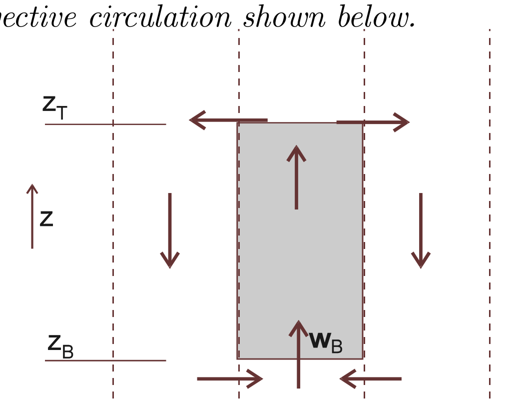
$$\left(\frac{p}{p_0}\right)^{\kappa} = -\frac{\kappa g}{RT_0}z + C$$

where C is a constant. Since $z = 0$ where $p = p_0$, $C = 1$ and so

$$\frac{p}{p_0} = \left[1 - \frac{\kappa g z}{RT_0}\right]^{\frac{1}{\kappa}} \ .$$

Thus, pressure vanishes at a height $z_0 = RT_0/\kappa g$. Note that z_0 is a factor $1/\kappa$ $(= 7/2)$ times the scale height based on surface temperature. Above z_0, the above solution is not valid (since there is no atmosphere there). Note also that $T \to 0$ there; in fact z_0 is exactly the height at which $T \to 0$ following the adiabatic lapse rate $-g/c_p$ from the surface.

11. *Consider the convective circulation shown below.*



Air rises in the center of the system; condensation occurs at altitude $z_B = 1km$ ($p_B = 880hPa$), and the convective cell (cloud is shown by *the shading) extends up to $z_T = 9km$ ($p_T = 330hPa$), at which point the air diverges and descends adiabatically in the downdraft region. The temperature at the condensation level, T_B, is $20^o C$. Assume all condensate falls out immediately as rain.*

(a) *Determine the specific humidity at an altitude of 3km within the cloud.*

(b) *The upward flux of air, per unit horizontal area, through the cloud at any level z is $w(z)\rho(z)$, where ρ is the density of dry air and w the vertical velocity. Mass balance requires that this flux be independent of height within the cloud. Consider the net upward flux of water vapor within the cloud and hence show that the rainfall rate below the cloud (in units of mass per unit area per unit time) is $w_B \rho_B [q_{*B} - q_{*T}]$, where the subscripts "$_B$" and "$_T$" denote the values at cloud base and cloud top, respectively. If $w_B = 5cm\ s^{-1}$, and $\rho_B = 1.0kg\ m^{-3}$, determine the rainfall rate in cm per day.*

a. Within the cloud, assume that air is saturated and the temperature therefore decreases with height at the saturated lapse rate 7K km^{-1}. At 3km altitude, 2km above cloud base, the temperature therefore is $20 - 14 = 6$°C. At 6°C, from Fig. 1.5, saturation vapor pressure is $e_s \simeq 10$hPa. To estimate pressure at this altitude, use

$$\frac{\partial p}{\partial z} = -g\rho = -g\frac{p}{RT} ,$$

and use the mean temperature $\bar{T}$ =13°C = 286K for the layer between the level of interest and cloud base. (Since T varies by no more than ±7K from this value, the error will be no more than 2.5%.) Then

$$\delta\left(\ln p\right) = \frac{\ln p(3\text{km})}{\ln p_B} = -\frac{g\ \delta z}{R\bar{T}}$$

where $\delta z = 2$km, so

$$\begin{aligned} p(3\text{km}) &= p_B \exp\left(-\frac{g\ \delta z}{R\bar{T}}\right) \\ &= 880 \times \exp\left(-\frac{9.81 \times 2000}{287 \times 286}\right) = 693\text{hPa} . \end{aligned}$$

Now, since the air is saturated, its specific humidity equals q_*, which is

$$q_* = \frac{R}{R_v}\frac{e_s}{p}$$

where R_v is the gas constant for water vapor, so the specific humidity at 3km is

$$q \simeq \frac{287}{462} \times \frac{10}{693} = 9.0 \text{ g kg}^{-1}.$$

b. The mass flux, $\mathcal{M}(z) = w\rho$, at altitude z, expresses the fact that a mass $\mathcal{M}$ of air crosses a unit cross-sectional area at altitude z per unit time. If the mixing ratio of water vapor (specific humidity) is q, then unit mass of air contains a mass q of water vapor. Therefore, the mass of water vapor crossing unit area per unit time is just $\mathcal{M}q = w\rho q$. Since the air is saturated everywhere within the cloud, the flux of water vapor within it is just $w\rho q_*$. Given that there is no mass flux out of the sides of the cloud, the net flux of water vapor into the cloud is

$$w_B\rho_B q_{*B} - w_T\rho_T q_{*T} \ .$$

However, the mass flux of dry air must (because of air mass conservation) be the same at all heights, so $w_T\rho_T = w_B\rho_B$, whence the net flux of water vapor into the cloud (mass of water vapor per unit horizontal area per unit time) is

$$F = w_B\rho_B\left(q_{*B} - q_{*T}\right) \ .$$

This will be positive (since q_* will decrease with altitude through the cloud), so there is a net flux of water vapor into the cloud, which must condense into liquid water. If all liquid water falls out as rain, then the rainfall rate, in terms of mass of liquid water per unit horizontal area per unit time, is just F.

Within the cloud, the air is saturated, so $\partial T/\partial z = -7\text{K km}^{-1}$. The temperature at cloud top, 8km above cloud base, is therefore $20 - 56 = -36°$C. From Fig. 1.5, we estimate the saturation vapor pressures as $e_{sB} = e_s(20\text{C}) \simeq 24$hPa; $e_{sT} = e_s(-36\text{C}) < 1$hPa. Since saturation specific humidity is

$$q_* = \frac{R}{R_v}\frac{e_s}{p} \ ,$$

we have

$$q_{*B} = \frac{287}{462} \times \frac{24}{880} = 0.0169 \; (= 16.9 \text{ g kg}^{-1})$$

and

$$q_{*T} < \frac{287}{462} \times \frac{1}{330} = 0.0019 \; (= 1.9 \text{ g kg}^{-1}) \; .$$

Thus, neglecting q_{*T} will incur a maximum error of a little over 10%. So the net rainfall rate is

$$\begin{aligned} r &= w_B \rho_B \left[q_{*B} - q_{*T} \right] \simeq w_B \rho_B q_{*B} \\ &= 0.05 \times 1.0 \times 0.0169 \\ &= 8.45 \times 10^{-4} \text{kg m}^{-2} \text{s}^{-1} \; . \end{aligned}$$

To convert this to more conventional units of depth per unit time, note that 1kg m^{-2} corresponds to a depth of $1/\rho_w = 10^{-3}$m of water, where $\rho_w = 10^3$kg m^{-3} is the density of liquid water. So

$$\begin{aligned} r &= \frac{8.45 \times 10^{-4}}{\rho_w} = 8.45 \times 10^{-7} \text{ms}^{-1} \\ &= 8.45 \times 10^{-7} \times 86400 = 7.30 \times 10^{-2} \text{ m day}^{-1} \\ &= 7.30 \text{ cm day}^{-1}. \end{aligned}$$

12. *Observations show that, over the Sahara, air continuously subsides (hence the Saharan climate). Consider an air parcel subsiding in this region, where the environmental temperature T_e decreases with altitude at the constant rate of $7K\ km^{-1}$.*

    (a) *Suppose the air parcel leaves height z with the environmental temperature. Assuming the displacement to be adiabatic, show that, after a time δt, the parcel is warmer than its environment by an amount $wS\ \delta t$, where w is the subsidence velocity and*

    $$S = \frac{dT_e}{dz} + \frac{g}{c_p} \; ,$$

    where c_p is the specific heat at constant pressure.

(b) *Suppose now that the displacement is not adiabatic, but that the parcel cools radiatively at such a rate that its temperature is always the same as its environment (so the circulation is in equilibrium). Show that the radiative rate of energy loss per unit volume must be $\rho c_p w S$, and hence that the net radiative loss to space, per unit horizontal area, must be*

$$\int_0^\infty \rho c_p w S \, dz \ = \frac{c_p}{g} \int_0^{p_s} w S \, dp \ ,$$

where p_s is surface pressure and ρ is the air density.

(c) *Radiative measurements show that, over the Sahara, energy is being lost to space at a net, annually-averaged rate of $20\,Wm^{-2}$. Estimate the vertically-averaged (and annually-averaged) subsidence velocity.*

(a) As the parcel descends adiabatically from height z (starting with the environmental temperature $T_e(z)$), its temperature will increase according to the dry adiabatic lapse rate $\Gamma_d = g/c_p$ (using our usual notation). [Note that, since the parcel will warm on descent, it will not saturate—even if it is saturated initially—so the dry value is the appropriate one.] After a time δt, the parcel will have descended a distance $\delta z = w \, \delta t$, where w is the subsidence velocity, and so its temperature will be

$$T_{parcel} = T_e(z) + \Gamma_d \, \delta z = T_e(z) + \frac{g}{c_p} w \, \delta t \ .$$

The environmental temperature at height $z - \delta z$ (given that the environmental lapse rate is linear in z) is

$$T_e(z - \delta z) = T_e(z) - \frac{dT_e}{dz} \delta z = T_e(z) - \frac{dT_e}{dz} w \, \delta t \ .$$

Therefore the temperature excess of the parcel is

$$\delta T = T_{parcel} - T_e(z - \delta z) = S w \, \delta t \ ,$$

where

$$S = \frac{dT_e}{dz} + \frac{g}{c_p} \ .$$

Note that S is a measure of the *static stability* of the environment—it expresses the departure of the atmospheric lapse rate from the adiabatic value. Since we are told that $dT_e/dz = -7\text{K km}^{-1}$ is constant, S will be constant also, $S = 3\text{K km}^{-1}$.

(b) In order for the parcel to maintain the environmental temperature, it must cool. Think of the sequence as a series of steps: adiabatic descent for a very short time δt, followed by non-adiabatic cooling to the environmental temperature (at constant pressure, since the parcel pressure must always equal that of the environment). After descending for a short time δt, the required heat loss per unit volume, δQ, is

$$\delta Q = \rho c_p \delta T = \rho c_p w S \ \delta t \ .$$

Therefore the heat loss per unit volume per unit time is

$$\frac{\delta Q}{\delta t} = \rho c_p w S \ .$$

Integrating in the vertical, and then using hydrostatic balance $dp = -g\rho \ dz$, the net cooling per unit horizontal area is

$$J_{cool} = \int_0^\infty \frac{\delta Q}{\delta t} dz = \int_0^\infty \rho c_p w S \ dz = \frac{c_p}{g} \int_0^{p_s} w S \ dp \ . \qquad (3)$$

QED. Note that the greater the stability S, or subsidence velocity w, the greater the cooling must be to sustain equilibrium.

(c) Observations show that, on the annual average, $J_{cool} = 20\text{Wm}^{-2}$. Since S is constant, we can rewrite (3) as

$$J_{cool} = \frac{c_p S}{g} p_s \langle w \rangle \ ,$$

where

$$\langle w \rangle = \frac{1}{p_s} \int_0^{p_s} w \ dp$$

is the vertically (pressure) averaged and annually-averaged subsidence velocity. Therefore

$$\begin{aligned} \langle w \rangle &= \frac{g J_{cool}}{c_p S p_s} \\ &= \frac{9.81 \times 20}{1005 \times 3 \times 10^{-3} \times 10^5} \ \text{ms}^{-1} \\ &= 0.65 \ \text{mm s}^{-1}. \end{aligned}$$