Atmosphere, Ocean and Climate Dynamics
Answers to Chapter 3

1. *Use the hydrostatic equation to show that the mass of a vertical column of air of unit cross-section, extending from the ground to great height, is $\frac{p_s}{g}$, where p_s is the surface pressure. Insert numbers to estimate the mass on a column of air of area $1\,\mathrm{m}^2$. Use your answer to estimate the total mass of the atmosphere.*

   From the hydrostatic relationship we can write (see Eq.3.4):

$$p_s = \int_{\text{surface}}^{\infty} \rho g dz$$

   Thus the mass of a column of air of area $1\,\mathrm{m}^2$ is the atmosphere is $\frac{p_s}{g} = \frac{10^5}{9.81} = 10^4\,\mathrm{kg}$. The total mass of the atmosphere is $M_a = \frac{4\pi a^2 \times p_s}{g} = \frac{4\pi \times \left(6.37\times 10^6\right)^2 \times 10^5}{9.81} = 5 \times 10^{18}\,\mathrm{kg}$.

2. *Using the hydrostatic equation, derive an expression for the pressure at the center of a planet in terms of its surface gravity, radius a and density ρ, assuming that the latter does not vary with depth. Insert values appropriate for the earth and evaluate the central pressure. [Hint: the gravity at radius r is $g(r) = \frac{Gm(r)}{r^2}$ where $m(r)$ is the mass inside a radius r and $G = 6.67 \times 10^{-11}\ \mathrm{kg}^{-1}\,\mathrm{m}^3\,\mathrm{s}^{-2}$ is the gravitational constant. You may assume the density of rock is $2000\,\mathrm{kg\,m}^{-3}$.]*

   The hydrostatic equation is

$$\frac{\partial p}{\partial r} = -\rho\left(r\right) g\left(r\right)$$

   where the acceleration due to gravity $g\left(r\right)$ at distance r is

$$g\left(r\right) = \frac{Gm\left(r\right)}{r^2}$$

   where $m\left(r\right)$ is the mass inside radius r. If the planet has a uniform density $m\left(r\right) = \frac{4}{3}\pi r^3 \rho$

$$\frac{\partial p}{\partial r} = -\rho G \frac{4}{3}\pi r^3 \frac{\rho}{r^2}$$

and so

$$\int_0^{p_o} dp = -\frac{4}{3}\pi\rho^2 G \int_a^0 r dr$$

giving

$$p_0 = \frac{2}{3}\pi G\rho^2 a^2$$

Noting that $M = \frac{4}{3}\pi a^3\rho$ is the mass of the planet and $g_a = \frac{MG}{a^2}$ is the surface gravity, then $p_0 = \frac{1}{2}\ g_a\rho a$.

Inserting numbers for the earth we find that: $p_0 = \frac{1}{2}\ 9.81 \times 2000 \times 6.37 \times 10^6 = 1.25 \times 10^{11} \simeq 10^6$ atmospheres.

3. *Consider a horizontally uniform atmosphere in hydrostatic balance. The atmosphere is isothermal, with temperature of* $-10^o C$. *Surface pressure is* $1000 hPa$.

   (a) *Consider the level that divides the atmosphere into two equal parts by mass (i.e., one-half of the atmospheric mass is above this level). What is the altitude, pressure, density and potential temperature at this level?*

   For an isothermal atmosphere in hydrostatic balance,

   $$\frac{\partial p}{\partial z} = -g\rho = -\frac{gp}{RT}$$

   whence

   $$p(z) = p(0)\exp\left(-\frac{z}{H}\right)$$

   where $H = RT/g$. With $R = 287\mathrm{J\,kg^{-1}\,K^{-1}}$, $T = -10\,^\circ\mathrm{C} = 263\,\mathrm{K}$, $H = 7.69\,\mathrm{km}$. Mass above level z is $M(z)$ where

   $$M(z) = \int_z^\infty \rho\, dz = -g\int_z^\infty \frac{\partial p}{\partial z}\, dz = gp(z)\ .$$

   The level at which $M(z) = M(0)/2$ has

   $$p(z) = p(0)/2 = 500\mathrm{hPa} = 5 \times 10^4\mathrm{Pa}.$$

At this level, the altitude is

$$z = -H \ln \left(\frac{p(z)}{p(0)} \right) = H \ln 2 = 7.69 \ln 2 = 5.33\,\text{km}.$$

Temperature is 263K, so density is

$$\rho = p/RT = 5 \times 10^4 / (287 \times 263) = 0.662\,\text{kg m}^{-3}.$$

(b) *Repeat the calculation of part (a) for the level below which lies 90% of the atmospheric mass.*

The level which has 90% of the mass below it has 10% above, so $M(z) = 0.1M(0)$, whence

$$p(z) = 0.1p(0) = 100\text{hPa} = 1 \times 10^4\text{Pa}.$$

The altitude is

$$z = -H \ln \left(\frac{p(z)}{p(0)} \right) = 7.69 \ln 10 = 17.71\,\text{km}.$$

Here, density is

$$\rho = p(z)/RT = 10^4 / (287 \times 263) = 0.132\,\text{kg m}^{-3}.$$

4. *Derive an expression for the hydrostatic atmospheric pressure at height z above the surface in terms of the surface pressure p_s and the surface temperature T_s for: (i) an isothermal atmosphere at temperature T_s and (ii) an atmosphere with constant lapse rate of temperature $\Gamma = -\frac{dT}{dz}$. Express your results in terms of the dry adiabatic lapse rate $\Gamma_d = \frac{g}{c_p}$.*

(i) In integrating the hydrostatic equation over a depth of ~10km in the atmosphere, g can be taken as constant. Thus we have:

$$\frac{\partial p}{\partial z} = \frac{pg}{RT}$$

giving

$$p(z) = p_s e^{-\left(\frac{gz}{RT_s}\right)} \text{ for an isothermal atmosphere}$$

and

(ii)

$$p(z) = p_s \left(\frac{T_s - \Gamma z}{T_s} \right)^{\frac{\gamma}{\gamma-1}\frac{g}{\Gamma c_p}}$$

where $\gamma = \frac{c_p}{c_v}$ for an atmosphere with constant lapse rate.

5. *Spectroscopic measurements show that a mass of water vapor of more than* 3 $\mathrm{kg\,m^{-2}}$ *in a column of atmosphere is opaque to the 'terrestrial' waveband. Given that water vapor typically has a density of* $10^{-2}\,\mathrm{kg\,m^{-3}}$ *at sea level (see Fig.3.1) and decays in the vertical like* $e^{-\left(\frac{z}{b}\right)}$*, where* z *is the height above the surface and* $b \sim 3$ km*, estimate at what height the atmosphere becomes transparent to terrestrial radiation.*

   By inspection of the observed vertical temperature profile shown in Fig.3.3, deduce the temperature of the atmosphere at this height. How does it compare to the emission temperature of the Earth, $T_e = 255\,\mathrm{K}$*, discussed in Chapter 2? Comment on your answer.*

   Writing the density of water vapor $\rho_v = \rho_{v_{surface}} e^{-\left(\frac{z}{b}\right)}$ then the atmosphere becomes transparent to terrestrial radiation at a height z^*, say, such that

$$\int_{z^*}^{\infty} \rho_v dz = \int_{z^*}^{\infty} \rho_{v_{surface}} e^{-\left(\frac{z}{b}\right)} dz = b\rho_{v_{surface}} e^{-\frac{z^*}{b}} = 3\,\mathrm{kg\,m^{-2}}$$

   This gives $z^* = 7\,\mathrm{km}$ if $b = 3\,\mathrm{km}$.

   The temperature at this height is $\sim 250\,\mathrm{K}$, roughly the same as the emission temperature of the planet. This provides excellent independent support to the idea that convective systems transport heat and moisture up to a height (and hence temperature) at which water vapor is the principle re-radiator of terrestrial wavelengths.

6. *Make use of your answer to Q.1 of Chapter 1 to estimate the error incurred in* p *at* $100\,\mathrm{km}$ *through use of Eq.(3.11) if a constant value of* g *is assumed.*

Assuming an isothermal atmosphere, Eq.(3.11) reduces to $p = p_s \exp(-z/H)$. Thus, given that $\frac{\partial p}{\partial H} = p_s \frac{z}{H^2} \exp(-z/H)$ then $\delta p = \frac{z}{H^2} p_s \exp(-z/H) \delta H$. Since $H = \frac{RT}{g}$ this implies that:

$$\frac{\delta p}{p} = \frac{z}{H} \frac{\delta g}{g}$$

Putting in numbers we find, noting that $\frac{\delta g}{g} = 3\%$ (from Q.1 of Chapter 1), $\frac{\delta p}{p} = \frac{100}{7.3} \times \frac{3}{100} = 0.41$ which is a large error. When pressures are being computed at many scale heights above the surface, large errors are incurred if variations in g are not accounted for.