Atmosphere, Ocean and Climate Dynamics
Answers to Chapter 2

1. *At present the emission temperature of the Earth is* $255K$*, and its albedo is* 30%*. How would the emission temperature change if*

   (a) *the albedo were reduced to* 10% *(and all else were held fixed);*

   (b) *the infra-red opacity of the atmosphere were doubled, but albedo remains fixed at* 30%*.*

   The emission temperature is defined as

   $$T_e = \left[\frac{(1-\alpha_p)S}{4\sigma}\right]^{\frac{1}{4}}, \tag{1}$$

   where α_p is the planetary albedo, S the solar flux, and σ the Stefan-Boltzmann constant.

   (a) If albedo were reduced from $\alpha_p = 30\%$ to $\alpha_p' = 10\%$, the emission temperature would change from T_e (at present) to T_e', where

   $$\frac{T_e'}{T_e} = \left[\frac{1-\alpha_p'}{1-\alpha_p}\right]^{\frac{1}{4}} = \left[\frac{0.9}{0.7}\right]^{\frac{1}{4}} = 1.0648 ,$$

   so the new emission temperature would be $255 \times 1.0648 = 271.5$K.

   (b) Emission temperature—the temperature at which the Earth emits to space—would not change at all if atmospheric IR opacity were doubled but albedo remained fixed. Emission temperature—unlike surface temperature—depends only on how much of the solar energy flux is absorbed by the Earth and, by (1), depends only on α_p, S, and σ.

2. *Suppose that the Earth is, after all, flat. Specifically, consider it to be a thin circular disk (of radius* $6370km$*), orbiting the Sun at the same distance as the Earth; the planetary albedo is 30%. The vector normal to one face of this disk always points directly toward the Sun, and the disk is made of perfectly conducting material, so both faces of the disk*

are at the same temperature. Calculate the emission temperature of this disk, and compare with the result we obtained for a spherical Earth.

Incoming solar flux $S_0 = 1367\text{Wm}^{-2}$; planetary albedo $\alpha_p = 0.3$. Area of disk intercepting solar flux= πa^2. So,

$$\text{Net solar input} = S_0 \pi a^2 (1 - \alpha_p) \, .$$

Disk has temperature on both faces, so area emitting thermal radiation is $2\pi a^2$. Disk emits σT_e^4 per unit area, so

$$\text{Net thermal emission } = 2\pi a^2 \sigma T_e^4 \, .$$

Balancing input and emission,

$$(1 - \alpha_p) S_0 \pi a^2 = 2\pi a^2 \sigma T_e^4 \, ,$$

i.e.,

$$T_e = \left[\frac{(1 - \alpha_p) S_0}{2\sigma} \right]^{\frac{1}{4}} = 303.1\text{K} \, .$$

The expression for T_e is a factor $2^{\frac{1}{4}}$ larger than we found for a spherical Earth—the disk has the same cross-section as the sphere (and so intercepts the same amount of solar radiation) but one-half of the surface area, so must increase T_e^4 by a factor of 2 to compensate.

3. *Consider the thermal balance of Jupiter. You will need the following information about Jupiter: mean planetary radius* $= 69500km$*; mean radius of orbit around the Sun* $= 5.19A.U.$ *(where* $1A.U.$ *is the mean radius of the Earth's orbit); planetary albedo*$= 0.51$*.*

   (a) *Assuming a balance between incoming and outgoing radiation, calculate the emission temperature for Jupiter.*

Solar flux at earth orbit $S_0 = 1367\text{Wm}^{-2}$, so solar flux at Jupiter's orbit is

$$S_J = S_0 \left(\frac{\text{mean radius of earth's orbit}}{\text{mean radius of Jupiter's orbit}} \right)^2 = \frac{1367}{(5.19)^2} = 50.75\text{Wm}^{-2} \, .$$

Given a Jupiter albedo $\alpha_J = 0.51$,

$$\text{net solar input} = S_J\,(1-\alpha_J)\,\pi a_J^2 = 3.77\times 10^{17} W.$$

Assuming blackbody radiation at temperature T_J,

$$\text{net thermal emission} = 4\pi a_J^2 \sigma T_J^4\ .$$

Assuming these balance gives

$$T_J = \left[\frac{(1-\alpha_J)S_J}{4\sigma}\right]^{\frac{1}{4}} = 102.3\text{K}\ .$$

b. *In fact, Jupiter has an internal heat source resulting from continued planetary contraction. Using the conventional definition of emission temperature T_e,*

$$\sigma T_e^4 = (\text{outgoing flux of planetary radiation per unit surface area})$$

the measured emission temperature of Jupiter is $130K$. Calculate the magnitude of Jupiter's internal heat source.

Observations show actual emission temperature is $T_J^{actual} = 130\text{K}$, i.e. Therefore, if the net internal heat source is H,

$$\begin{aligned} H &= (\text{net thermal emission}) - (\text{net solar input}) \\ &= \pi a_J^2 \left[4\sigma \left(T_J^{actual}\right)^4 - S_J\,(1-\alpha_J)\right] \\ &= 6.06\times 10^{17}\text{W}. \end{aligned}$$

c. *It is believed that the source of Q on Jupiter is the release of gravitational potential energy by a slow contraction of the planet. On the simplest assumption that Jupiter is of uniform density and remains so as it contracts, calculate the annual change in its radius a_{jup} required to produce your value of Q. (Only one half of the released gravitational energy is convertible to heat, the remainder appearing as the additional kinetic energy required to preserve the angular momentum of the planet.)*

[A uniform sphere of mass M and radius a has a gravitational potential energy of $-\frac{3}{5}G\frac{M^2}{a}$ where G is the gravitational constant $= 6.7 \times 10^{-11}\,\mathrm{kg^{-1}\,m^3\,s^{-2}}$. The mass of Jupiter is $2 \times 10^{27}\,\mathrm{kg}$ and its radius is $a_{jup} = 7.1 \times 10^7\,\mathrm{m}$.]

Expressing what we are told in mathematics we have:

$$\frac{1}{2}\frac{\partial}{\partial t}\left(-\frac{3}{5}G\frac{M^2}{a}\right) = 4\pi a^2 Q = H$$

and so, noting that $\frac{\partial}{\partial t}\left(-\frac{1}{a}\right) = \frac{1}{a^2}\frac{\partial}{\partial t}a$ and rearranging we find

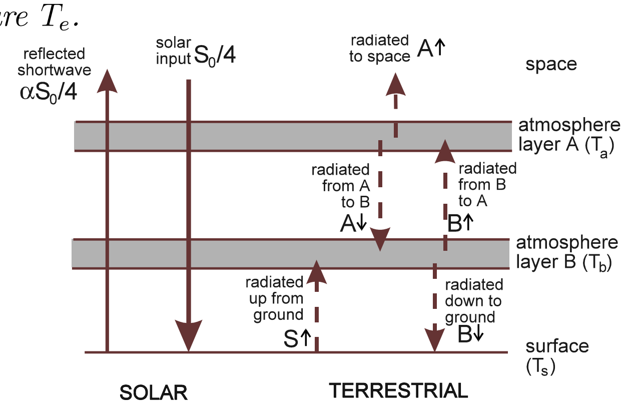
$$\frac{\partial}{\partial t}a = \frac{40\pi}{3}\frac{a^2}{GM^2}H.$$

Inserting numbers we have

$$\begin{aligned}\frac{\partial}{\partial t}a &= \frac{40\pi}{3}\frac{(7.1 \times 10^7\,\mathrm{m})^2}{6.7 \times 10^{-11}\,\mathrm{kg^{-1}\,m^3\,s^{-2}} \times (2 \times 10^{27}\,\mathrm{kg})^2} \times 6.06 \times 10^{17}\,\mathrm{J\,s^{-1}} \\ &= 4.8 \times 10^{-10}\,\mathrm{m\,s^{-1}} = 1.5 \times 10^{-2}\,\mathrm{m} \text{ per year!}\end{aligned}$$

4. *Consider the "two-slab" greenhouse model illustrated in the figure below in which the atmosphere is represented by two perfectly absorbing layers of temperature T_a and T_b.*

   Determine T_a, T_b, and the surface temperature T_s in terms of the emission temperature T_e.



Each layer (and the surface) radiates as a blackbody, and is totally absorbing. Therefore fluxes (per unit area) are

$$\begin{aligned} A &\uparrow = A \downarrow = \sigma T_a^4 \ ; \\ B &\uparrow = B \downarrow = \sigma T_b^4 \ ; \\ S &\uparrow = \sigma T_s^4 \ . \end{aligned}$$

Net input per unit area from space to the Earth-atmosphere system is $(1 - \alpha_p) S_0/4$. Net output per unit area is σT_a^4. Therefore

$$\begin{aligned} \sigma T_a^4 &= (1 - \alpha_p) \frac{S_0}{4} \ , \ \text{or} \\ T_a &= \left[(1 - \alpha_p) \frac{S_0}{4\sigma} \right]^{\frac{1}{4}} = T_e \ . \end{aligned}$$

Consider balance of layer A. Only inputs and outputs are IR; net input is $B \uparrow$; net output is $A \uparrow + A \downarrow$. So

$$B \uparrow = \sigma T_b^4 = A \uparrow + A \downarrow = 2\sigma T_a^4 = 2\sigma T_e^4 \ ,$$

so

$$T_b = 2^{\frac{1}{4}} T_e \ .$$

Consider balance of layer B. Net input is $A \downarrow + S \uparrow$; net output is $B \uparrow + B \downarrow$. Therefore

$$S \uparrow = B \uparrow + B \downarrow - A \downarrow \ ,$$

i.e.,

$$\begin{aligned} \sigma T_s^4 &= 4\sigma T_a^4 - \sigma T_e^4 \\ &= 3\sigma T_e^4 \ . \end{aligned}$$

So

$$T_s = 3^{\frac{1}{4}} T_e \ .$$

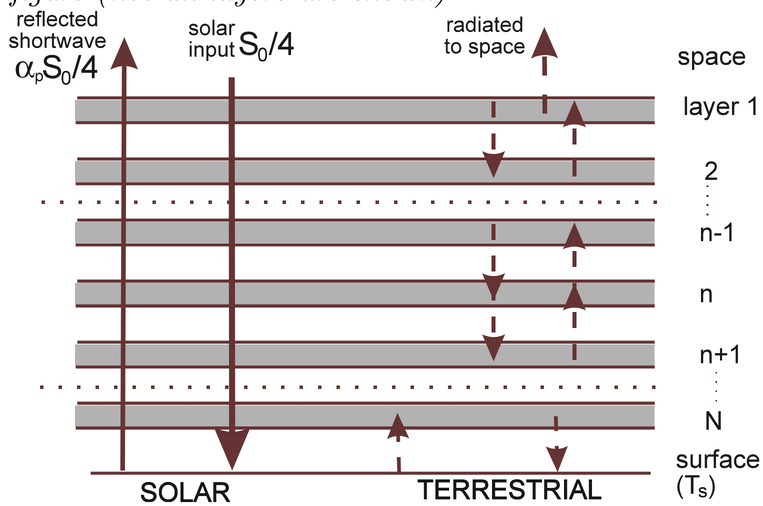
QED. Can also (as a check) consider surface balance. Net input there is $(1 - \alpha_p) S_0/4 + B \downarrow$; net output is $S \uparrow$. With the above results,

$$\begin{aligned} \text{net input} &= (1 - \alpha_p) S_0/4 + B \downarrow = \sigma T_e^4 + \sigma T_b^4 \\ &= 3\sigma T_e^4 \ ; \end{aligned}$$

$$\begin{aligned} \text{net output} &= S \uparrow = \sigma T_s^4 \\ &= 3\sigma T_e^4 \ . \end{aligned}$$

These are balanced, as they should be.

5. *Consider an atmosphere that is completely transparent to shortwave (solar) radiation, but very opaque to infrared (IR) terrestrial radiation. Specifically, assume that it can be represented by N slabs of atmosphere, each of which is completely absorbing of IR, as depicted in the following schematic figure (not all layers are shown).*



Assume blackbody radiation. Then each atmospheric layer radiates both up and down with a flux σT_n^4 per unit area. The upward flux from the surface is σT_s^4. The net solar flux per unit area, which is absorbed only at the surface, is $\frac{1}{4}S_0(1-\alpha_p) = \sigma T_e^4$, where T_e is the equilibrium temperature.

(a) *By considering the radiative equilibrium of the surface, show that the surface must be warmer than the lowest atmospheric layer.*

Because IR radiation reaching the surface comes only from the N^{th} layer, the surface heat budget, in equilibrium, is

$$\text{net solar input} + \text{IR from layer } N = \text{net IR loss from surface}$$

i.e.,

$$\begin{aligned} \sigma T_e^4 + \sigma T_N^4 &= \sigma T_s^4 \text{ , or} \qquad (6.1) \\ T_s^4 &= T_N^4 + T_e^4 \, . \end{aligned}$$

Therefore $T_s > T_N$ —the surface is necessarily warmer than the lowest atmospheric layer. This is a simple consequence of the fact that the surface is heated by solar radiation as well as downwelling IR from the lowest layer. (If no solar radiation reaches the surface, $T_s = T_N$.)

(b) *By considering the radiative equilibrium of the n^{th} layer, show that, in equilibrium,*

$$2T_n^4 = T_{n+1}^4 + T_{n-1}^4 \ ,$$

where T_n is the temperature of the n^{th} layer, for $n > 1$. Hence argue that the equilibrium surface temperature is

$$T_s = (N+1)^{\frac{1}{4}}\, T_e \ ,$$

where T_e is the planetary emission temperature.

Consider layer $n > 1$. This layer loses heat by radiating IR both up and down, so its net rate of heat loss per unit area is $2\sigma T_n^4$. It receives IR from the layer above (σT_{n-1}^4 per unit area) and below (σT_{n+1}^4 per unit area). balancing input and output, therefore,

$$2T_n^4 = T_{n-1}^4 + T_{n+1}^4 \ . \qquad (6.2)$$

Note that this result is valid for all $n > 1$, including $n = N$, in which case layer $n+1$ is the surface.

Now consider the net radiation between the Earth and space, which must be zero in equilibrium. Net input per unit area from the Sun is $\frac{1}{4}S_0\,(1-\alpha_p) = \sigma T_e^4$; the output to space per unit area, from the top of the atmosphere, is σT_1^4. Balancing these,

$$T_1 = T_e \ , \qquad (6.3)$$

Thus defining T_1, the temperature of the top layer. If we consider the balance of the top layer itself, its loss of heat per unit area (via IR up and down) is $2\sigma T_1^4$, while the gain is from layer 2 only, and is σT_2^4 per unit area. Equating these,

$$T_2^4 = 2T_1^4 = 2T_e^4. \tag{6.4}$$

Now, (6.2) gives

$$T_{n+1}^4 - T_n^4 = T_n^4 - T_{n-1}^4 \tag{6.5}$$

so the difference in T^4 between adjacent layers is the same. Given (6.4) at the top layer, it follows from (6.5) that $T_{n+1}^4 - T_n^4 = T_e^4$ for all n. From (6.3), it then follows that

$$T_n^4 = nT_e^4 .$$

With $n = N + 1$ at the surface, then, the surface temperature is

$$T_s = (N+1)^{\frac{1}{4}} T_e .$$

6. *Determine the emission temperature of the planet Venus. You may assume the following: the mean radius of Venus' orbit is* 0.72 *times that of the Earth's orbit. Given the solar flux decreases like the square of the distance from the sun and given that the planetary albedo of Venus* $= 0.77$, *determine the emission temperature of Venus.*

   According to the inverse square law,

$$\frac{\text{solar flux at Venus}}{\text{solar flux at Earth}} = \left(\frac{\text{radius of Venus orbit}}{\text{radius of Earth orbit}}\right)^{-2}$$

   whence

$$\text{solar flux at Venus} = S_V = \frac{1367}{0.72^2} = 2637\text{Wm}^{-2}.$$

   Therefore, in equilibrium, the emission temperature of Venus is

$$T_V = \left[\frac{S_V(1-\alpha_V)}{4\sigma}\right]^{\frac{1}{4}},$$

   where $\alpha_V = 0.77$ is the planetary albedo of Venus and σ the Stefan-Boltzmann constant. Therefore

$$T_V = \left[\frac{2637 \times 0.23}{4 \times 5.67 \times 10^{-8}}\right]^{\frac{1}{4}} = 227\text{K}.$$

The observed mean surface temperature of the planet Venus is about 750K. This is much greater than the emission temperature of Venus calculated in Q.3. How many layers of the $N-$layer model considered in Q5 would be required to achieve this degree of warming? Comment.

The observation that the surface temperature is about 750K suggests an extremely efficient greenhouse effect on Venus. (In terms of the N-layer model, we would need $(750/227)^4 - 1 \cong 100$ layers to achieve this degree of warming.) Indeed Venus has a much thicker atmosphere than the Earth (surface pressure on Venus $\simeq$ 90 times that on the Earth), which consists mostly of CO_2, so it is extremely opaque to IR.

7. *Climate feedback due to Stefan-Boltzmann.*

   (a) *Show that the globally-averaged incident solar flux at the ground is $\frac{1}{4}(1-\alpha_p)S_0$.*

   The solar radiation reaching the surface over the globe is (see Fig.2.4) $(1-\alpha_p)S_0\pi a^2$. To obtain the global average, we divide by the surface area $4\pi a^2$ to yield a globally-averaged incident solar flux at the ground of $\frac{1}{4}(1-\alpha_p)S_0$.

   (b) *If the outgoing longwave radiation from the earth's surface were governed by the Stefan-Boltzmann law, then we showed in Eq.(2.15) that for every $1\,W\,\mathrm{m}^{-2}$ increase in the forcing of the surface energy balance, the surface temperature will increase by about a quarter of a degree. Use your answer to (a) to estimate by how much one would have to increase the solar constant to achieve a 1 °C increase in surface temperature? You may assume that the albedo of earth is 0.3 and does not change.*

   To achieve a $1\,^\circ\mathrm{C}$ rise in surface temperature we require a $4\,W\,\mathrm{m}^{-2}$ increase in the forcing of the surface. Thus $\frac{\delta S_0(1-\alpha_p)}{4} = 4\,W\,\mathrm{m}^{-2}$ implying that $\delta S_0 = \frac{16}{1-0.3} = 22\,W\,\mathrm{m}^{-2}$. This is 1.6% of the solar constant, a significant increase.