Chapter 13

Numerical Modeling and Prediction

13.1. Show that for the barotropic vorticity equation on the Cartesian β-plane (13.26) enstrophy and kinetic energy are conserved when averaged over the whole domain—that is, that the following integral constraints are satisfied:

$$\frac{d}{dt}\iint \frac{\zeta^2}{2}dxdy = 0 \qquad \frac{d}{dt}\iint \frac{\nabla\psi\cdot\nabla\psi}{2}dxdy = 0$$

Hint: To prove energy conservation, multiply (13.26) through by $-\psi$ and use the chain rule of differentiation.

Solution: Here we may use periodic boundary conditions in x and let ψ = constant at $y = 0$ and $y = D$. For enstrophy conservation take ζ times (13.26):

$$\frac{1}{2}\frac{\partial\zeta^2}{\partial t} = -\mathbf{V}_\psi\cdot\nabla\left(\frac{\zeta^2}{2}\right) - \beta\zeta\frac{\partial\psi}{\partial x} = \underset{(1)}{-\nabla\cdot\left(\frac{\mathbf{V}_\psi\zeta^2}{2}\right)} - \beta\frac{\partial\psi}{\partial x}\left(\underset{(2)}{\frac{\partial^2\psi}{\partial x^2}} + \underset{(3)}{\frac{\partial^2\psi}{\partial y^2}}\right)$$

Integrating term (1) over the area of the domain and invoking periodicity in x yields $\left(v_\psi\zeta^2/2\right)\Big|_0^D = 0$, since $v_\psi = \partial\psi/\partial x$, which vanishes at the y boundaries. Term (2) can be written as $-\frac{\beta}{2}\frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial x}\right)^2$, which vanishes when integrated in x. Term (3) can be expanded to $\frac{\beta}{2}\frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial y}\right)^2 - \beta\frac{\partial}{\partial y}\left(\frac{\partial\psi}{\partial x}\frac{\partial\psi}{\partial y}\right)$. The first part vanishes when integrated in x and the second part after integration in y.

For energy conservation, the left-hand side gives the rate of change of kinetic energy: $-\psi\frac{\partial\nabla^2\psi}{\partial t} = -\nabla\cdot\left(\psi\nabla\frac{\partial\psi}{\partial t}\right) + \nabla\psi\cdot\nabla\frac{\partial\psi}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\nabla\psi\cdot\nabla\psi}{2}\right) = \frac{\partial}{\partial t}\left(\frac{\mathbf{V}_\psi\cdot\mathbf{V}_\psi}{2}\right)$, where the term $\nabla\cdot\left(\psi\nabla\frac{\partial\psi}{\partial t}\right)$ vanishes after integration in x and y and application of the boundary conditions.

The right-hand side gives $-\psi\mathbf{V}_\psi\cdot\nabla\zeta = -\nabla\cdot\left(\mathbf{V}_\psi\psi\zeta\right) + \zeta\psi\nabla\cdot\mathbf{V}_\psi + \mathbf{V}_\psi\zeta\cdot\nabla\psi$. The first term on the right vanishes after integration over the domain. The second vanishes because $\mathbf{V}_\psi$ is nondivergent. The third vanishes because $\mathbf{V}_\psi$ is orthogonal to $\nabla\psi$. Finally, the term $\beta\psi\frac{\partial\psi}{\partial x} = \frac{\beta}{2}\frac{\partial\psi^2}{\partial x}$ vanishes when integrated over x. Thus, both enstrophy and kinetic energy are conserved on the domain.

13.2. Verify expression (13.31). Use periodic boundary conditions in both x and y.

Solution: $\sum\limits_m\sum\limits_n F_{m,n} = 1/2d\sum\limits_m\sum\limits_n\left(u_{m+1,n}\zeta_{m+1,n} - u_{m-1,n}\zeta_{m-1,n}\right)$

$$+1/2d\sum_m\sum_n\left(v_{m,n+1}\zeta_{m,n+1} - v_{m,n-1}\zeta_{m,n-1}\right) + \beta/2d\sum_m\sum_n\left(\psi_{m+1,n} - \psi_{m-1,n}\right)$$

Each of the terms on the right consists of a pair of opposite signs with m or n indices differing by 2. Thus, it is clear that alternate terms cancel in the summation for all m, n provided that boundary conditions are periodic.

13.3. The Euler backward method of finite differencing the advection equation is a two-step method consisting of a forward prediction step followed by a backward corrector step. In the notation of Section 13.2.2, the complete cycle is thus defined by

$$\hat{q}^*_m - \hat{q}_{m,s} = -\frac{\sigma}{2}\left(\hat{q}_{m+1,s} - \hat{q}_{m-1,s}\right)$$
$$\hat{q}_{m,s+1} - \hat{q}_{m,s} = -\frac{\sigma}{2}\left(\hat{q}^*_{m+1} - \hat{q}^*_{m-1}\right),$$

where $\hat{q}^*_m$ is the first guess for time step $s+1$. Use the method of Section 13.2.3 to determine the necessary condition for stability of this method.

Solution: Let $\hat{q}^*_m = A^s \exp(ipm)$; $\hat{q}_{m,s} = B^s \exp(ipm)$. Then substituting into the above equations gives $A^s - B^s = -B^s(\sigma i \sin p)$; $B^{s+1} - B^s = -A^s(\sigma i \sin p)$, where we have used the fact that $2i \sin p = \exp(ip) + \exp(-ip)$. Thus, eliminating A^s, we get $B^{s+1} = B^s[1 - (\sigma i \sin p)(1 - \sigma i \sin p)]$. Letting $\mu = \sigma \sin p$, we obtain $B = 1 - i\mu(1 - i\mu) = 1 - i\mu - \mu^2$. But $|B| = |1 - i\mu - \mu^2| \le 1$ for stability, which requires $\mu^2 \le 1$, so $\sigma \le 1$.

13.4. Carry out truncation error analyses analogous to that of Table 13.1 for the centered difference approximation to the advection equation but for the cases $\sigma = 0.95$ and $\sigma = 0.25$.

Solution: For $\sigma = 0.95$

$L/\delta x$	p	θ_p	c'/c	$\|D\|/\|C\|$
2	π	π	–	∞
4	$\pi/2$	1.253	0.840	0.524
8	$\pi/4$	0.737	0.988	0.149
16	$\pi/8$	0.372	0.997	0.035
32	$\pi/16$	0.186	0.997	0.009

For $\sigma = 0.25$

$L/\delta x$	p	θ_p	c'/c	$\|D\|/\|C\|$
2	π	π	–	∞
4	$\pi/2$	0.253	0.644	0.016
8	$\pi/4$	0.178	0.907	0.008
16	$\pi/8$	0.096	0.976	0.002
32	$\pi/16$	0.049	0.994	0.001

13.5. Suppose that the streamfunction ψ is given by a single sinusoidal wave $\psi(x) = A\sin(kx)$. Find an expression for the error of the finite difference approximation $\partial^2\psi/\partial x^2 \approx (\psi_{m+1} - 2\psi_m + \psi_{m-1})/(\delta x)^2$ for $k\delta x = \pi/8, \pi/4, \pi/2$, and π. Here $x = m\delta x$ with $m = 0, 1, 2, \ldots$.

Solution: $\partial^2\psi/\partial x^2 = -Ak^2 \sin kx = -Ak^2 \sin km\delta x$, but

$$\begin{aligned}\psi_{m+1} - 2\psi_m + \psi_{m-1} &= A[\sin k(m+1)\delta x - 2\sin km\delta x + \sin k(m-1)\delta x] \\ &= 2A \sin mk\delta x(\cos k\delta x - 1).\end{aligned}$$

Thus, the fractional error $= 1 - [2A \sin mk\delta x(1 - \cos k\delta x)(\delta x)^{-2}](Ak^2 \sin km\delta x)^{-1} = 1 - 2k^{-2}(\delta x)^{-2}(1 - \cos k\delta x)$, which gives errors of 0.0128, 0.050, 0.189, and 0.595 for $k\delta x = \pi/8, \pi/4, \pi/2$, and π, respectively.

13.6. Using the method given in Section 13.2.3, evaluate the computational stability of the following two finite difference approximations to the one-dimensional advection equation:

(a) $$\hat{\zeta}_{m,s+1} - \hat{\zeta}_{m,s} = -\sigma\left(\hat{\zeta}_{m,s} - \hat{\zeta}_{m-1,s}\right)$$

(b) $$\hat{\zeta}_{m,s+1} - \hat{\zeta}_{m,s} = -\sigma\left(\hat{\zeta}_{m+1,s} - \hat{\zeta}_{m,s}\right)$$

where $\sigma = c\delta t/\delta x > 0$. (The schemes labeled (a) and (b) are referred to as *upstream* and *downstream* differencing, respectively.) Show that scheme (a) damps the advected field, and compute the fractional damping rate per time step for $\sigma = 0.25$ and $k\delta x = \pi/8$ for a field with the initial form $\zeta = \exp(ikx)$.

Solution: Let $\hat{\zeta}_{m,r} = V_r \exp(ikmd)$, where $d = \delta x$. Then upstream differencing gives $V_{r+1} - V_r = -\sigma V_r(1 - e^{-ikd})$, $V_{r+1} = V_r[1 - \sigma(1 - e^{-ikd})]$, which implies that $|1 - \sigma(1 - e^{-ikd})| \le 1$ for stability. Thus, $[1 - \sigma(1 - \cos kd)]^2 +$

$\sigma^2(\sin kd)^2 \leq 1$ or $1 - 2\sigma(1-\cos kd) + \sigma^2(1+\cos^2 kd + \sin^2 kd - 2\cos kd) = 1 - 2\sigma(1-\cos kd)(1-\sigma) \leq 1$, which implies that $\sigma \leq 1$ for stability.

Downstream differencing gives $V_{r+1} = V_r[1 - \sigma(e^{ikd} - 1)]$, which implies that we need $|1 - \sigma(e^{ikd} - 1)|^2 = (1 - \sigma\cos kd + \sigma)^2 + (\sigma \sin kd)^2 \leq 1$ for stability. Thus, $1 + 2\sigma(1-\cos kd) + \sigma^2(1-\cos kd)^2 + \sigma^2(\sin kd)^2 \leq 1$. But this expression is false for all real values of σ. Thus, the scheme is absolutely unstable.

The fractional damping rate for upstream differencing is given by $|V_{r+1}/V_r| = |1 - \sigma(1 - e^{-ikd})|$. For $kd = \pi/8$ and $\sigma = 0.25$ this gives 0.985, so that amplitude decreases by the fraction $1 - 0.985 = 0.015$ per time step.

13.7. Using a staggered horizontal grid analogous to that shown in Figure 13.5 (but for an equatorial β-plane geometry) express the linearized shallow water equations (11.27)–(11.29) in finite difference form.

Solution: The staggered grid system has the following arrangement:

$V_{m-1,n+1}$		$V_{m,n+1}$		$V_{m+1,n+1}$
$\Phi_{m-1,n}$	$U_{m,n}$	$\Phi_{m,n}$	$U_{m+1,n}$	$\Phi_{m+1,n}$
$V_{m-1,n}$		$V_{m,n}$		$V_{m+1,n}$
$\Phi_{m-1,n-1}$	$U_{m,n-1}$	$\Phi_{m,n-1}$	$U_{m+1,n-1}$	$\Phi_{m+1,n-1}$
$V_{m-1,n-1}$		$V_{m,n-1}$		$V_{m+1,n-1}$

Define mean and difference fields as follows for all variables:

$$\overline{G}^x(m+1/2, n) \equiv \frac{1}{2}[G(m+1,n) + G(m,n)]$$
$$\delta_x G(m+1/2, n) \equiv \frac{1}{\delta x}[G(m+1,n) - G(m,n)]$$

(with analogous definitions for $\overline{G}^y$ and $\delta_y G$). Then (11.29)–(11.31) can be written as

$$\delta_t u(m,n) = \beta y_n \overline{\overline{v}^x}^y(m-1/2, n+1/2) - \delta_x \Phi(m-1/2, n)$$
$$\delta_t v(m,n) = -\beta y_{n-1/2} \overline{\overline{u}^x}^y(m+1/2, n-1/2) - \delta_y \Phi(m, n-1/2)$$
$$\delta_t \Phi(m,n) = -gh_e[\delta_x u(m+1/2, n) + \delta_y v(m, n+1/2)]$$

(where $\delta_t G$ stands for a suitable time differencing scheme).

13.8. Verify the equality

$$\left(\frac{1 - i\tan\theta_p}{1 + i\tan\theta_p}\right) = \exp(-2i\theta_p)$$

given in (13.22).

Solution:

$$\frac{1 - i\tan\theta}{1 + i\tan\theta} = \frac{\cos\theta - i\sin\theta}{\cos\theta + i\sin\theta} = \frac{e^{-i\theta}}{e^{+i\theta}} = e^{-2i\theta}$$

13.9. Compute the ratio of the numerical phase speed to the true phase speed, c'/c, for the implicit differencing scheme of (13.19) for $p = \pi, \pi/2, \pi/4, \pi/8$, and $\pi/16$. Let $\sigma = 0.75$ and $\sigma = 1.25$. Compare your results to those of Table 13.1.

Solution: Now, $\theta_p = \tan^{-1}[\sigma(\sin p)/2]$, and from below (13.23),

$$c'/c = (2\theta_p)/(ck\delta t) = (2\theta_p)/(\sigma p).$$

		$\sigma = 0.75$		$\sigma = 1.25$	
$L/\delta x$	p	θ_p	c'/c	θ_p	c'/c
2	π	π	–	π	–
4	$\pi/2$	0.359	0.609	0.559	0.569
8	$\pi/4$	0.259	0.880	0.416	0.848
16	$\pi/8$	0.143	0.968	0.235	0.957
32	$\pi/16$	0.073	0.992	0.121	0.989

13.10. Using the technique of Section 13.2.1, show that the following 4-point difference formula is of fourth order accuracy:

$$\psi'(x_0) \approx \frac{4}{3}\left(\frac{\psi(x_0+\delta x)-\psi(x_0-\delta x)}{2\delta x}\right) - \frac{1}{3}\left(\frac{\psi(x_0+2\delta x)-\psi(x_0-2\delta x)}{4\delta x}\right)$$

Solution: Subtracting (13.3) from (13.2) gives

$$\psi(x_0+\delta x)-\psi(x_0-\delta x) = \psi'(x_0)(2\delta x) + \psi'''(x_0)\frac{(\delta x)^3}{3} + \mathrm{O}\left[(\delta x)^5\right], \tag{1}$$

while applying the same formula for the interval $2\delta x$ yields

$$\psi(x_0+2\delta x)-\psi(x_0-2\delta x) = \psi'(x_0)(4\delta x) + \psi'''(x_0)\frac{(2\delta x)^3}{3} + \mathrm{O}\left[(2\delta x)^5\right]. \tag{2}$$

Now taking 1/6 of (2) plus 4/3 of (1) and solving for $\psi'(x_0)$ gives

$$\psi'(x_0) = \frac{4}{3}\left[\frac{\psi(x_0+\delta x)-\psi(x_0-\delta x)}{2\delta x}\right] - \frac{1}{3}\left[\frac{\psi(x_0+2\delta x)-\psi(x_0-2\delta x)}{4\delta x}\right] + \mathrm{O}\left[(\delta x)^4\right],$$

which is the desired result.

13.11. The Dufort-Frankel method for approximating the one-dimensional diffusion equation

$$\frac{\partial q}{\partial t} = K\frac{\partial^2 q}{\partial x^2}$$

can be expressed in the notation of Section 13.2.2 as

$$\hat{q}_{m,s+1} = \hat{q}_{m,s-1} + r[\hat{q}_{m+1,s} - (\hat{q}_{m,s+1} + \hat{q}_{m,s-1}) + \hat{q}_{m-1,s}],$$

where $r \equiv 2K\delta t/(\delta x)^2$. Show that this scheme is an explicit differencing scheme and that it is computationally stable for all values of δt.

Solution: By rearranging terms, the above difference equation can be expressed as $(1+r)\hat{q}_{m,s+1} = (1-r)\hat{q}_{m,s-1} + r[\hat{q}_{m+1,s} + \hat{q}_{m-1,s}]$. This is an explicit algorithm, since $\hat{q}_{m,s+1}$ depends only on values at times s and $s-1$. To examine stability, we let $\hat{q}_{m,s} = B^s \exp(ipm)$, and substitute into the difference equation to get
$(1+r)B^2 = (1-r) + Br(e^{ip} + e^{-ip})$, so that $B^2 - \frac{(2r\cos p)}{(1+r)}B - \frac{(1-r)}{(1+r)} = 0$.
Thus, $B = \left(\frac{r\cos p}{1+r}\right) \pm \frac{1}{(1+r)}[1 - r^2\sin^2 p]^{1/2}$.
The largest magnitude of B occurs for $p = \pi$, in which case $B = -1$, so the system is unconditionally computationally stable.