Middle Atmosphere Dynamics

12.1. Suppose that temperature increases linearly with height in the layer between 20 and 50 km at a rate of $2\,\mathrm{K\,km^{-1}}$. If the temperature is 200 K at 20 km, find the value of the scale height H for which the log-pressure height z coincides with actual height at 50 km. (Assume that z coincides with the actual height at 20 km and let g be a constant.)

Solution: Recalling that $d\Phi = gdz$, we get integrating equation (10.3):
$\int_{z_0}^{z_T} \frac{gdz}{T} = \left(\frac{R}{H}\right) \int_{z_0^*}^{z_T^*} Dz^*$, where $z_0 = z_0^* = 20\,\mathrm{km}$ and $T = T_0 + \gamma(z - z_0)$ with $\gamma = 2 \times 10^{-3}\,\mathrm{K\,m^{-1}}$. Thus, $\frac{R}{H}\left(z_T^* - z_0^*\right) = \frac{g}{\gamma} \ln\left[\frac{T_0+\gamma(z-z_0)}{T_0}\right]$, so to have $z_T^* - z_0^* = 30\,\mathrm{km}$, we require $H = \frac{R(z_T^* - z_0^*)\gamma}{g \ln\left[\frac{T_0+\gamma(z-z_0)}{T_0}\right]} = \frac{287 \times 30 \times 2}{9.8 \times \ln\left(\frac{260}{200}\right)} = 6697\,\mathrm{m}$. (This scale height corresponds to a mean temperature of $T = 229\,\mathrm{K}$.)

12.2. Find the Rossby critical velocities for zonal wave numbers 1, 2, and 3 (i.e., for 1, 2, and 3 wavelengths around a latitude circle). Let the motion be referred to a β-plane centered at 45°N, scale height $H = 7$ km, buoyancy frequency $N = 2 \times 10^{-2}\,\mathrm{s^{-1}}$, and infinite meridional scale ($l = 0$).

Solution: From (12.16) $U_c = \beta\left[k^2 + f_0^2/\left(4N^2H^2\right)\right]^{-1}$ where $l = 0$. But here $k = s/(a\cos\phi)$ and $\beta = 2\Omega\cos\phi/a$. Thus, $U_c = 1.619 \times 10^{-11}\left[s^2\left(4.93 \times 10^{-14}\right) + 1.356 \times 10^{-13}\right]^{-1}$. Then,

s	U_c (m/s)
1	87.5
2	48.7
3	28.0

12.3. Suppose that a stationary linear Rossby wave is forced by flow over sinusoidal topography with height $h(x) = h_0\cos(kx)$, where h_0 is a constant and k is the zonal wavenumber. Show that the lower boundary condition on the streamfunction ψ can be expressed in this case as

$$\left(\partial\psi/\partial z\right) = -hN^2/f_0.$$

Using this boundary condition and an appropriate upper boundary condition, solve for $\psi(x,z)$ in the case $|m| \gg (1/2H)$ using the equations of Section 12.3.1. How does the position of the trough relative to the mountain ridge depend on the sign of m^2?

Solution: Now for topographic forcing $w' = \overline{u}\partial h/\partial x$ at $z = 0$. From the linearized form of the adiabatic thermodynamic energy equation [see (8.46)], we then have at $z = 0$:
$\left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right)\frac{\partial\Phi'}{\partial z} = -w'N^2 = -N^2\overline{u}\frac{\partial h}{\partial x}$. But $\Phi' = f_0\psi'$, so the lower boundary condition can be expressed for stationary waves $\left(\partial/\partial t = 0\right)$ as:
$\frac{\partial\psi'}{\partial z} = -\frac{N^2h}{f_0} = -\frac{N^2h_0}{f_0}\cos(kx)$. But from (12.13),
$\psi'(x,z) = \mathrm{Re}\{A\exp\left[i(kx+mz) + z/2H\right]\}$ where A is a complex constant. And m is given by (12.15). The lower boundary condition then permits determination of A:
$\frac{\partial\psi'}{\partial z} = \mathrm{Re}\left\{\left(im + \frac{1}{2H}\right)A\exp(ikx)\right\} = -\frac{N^2h_0}{f_0}\cos(kx)$, at $z = 0$. Thus, for $m^2 > 0$ and $|m| \gg 1/2H$, $\psi'(x,z) = -\frac{N^2h_0}{mf_0}\sin(kx+mz)\exp\left(\frac{z}{2H}\right)$.
For $m^2 < 0$ and $|m| \gg 1/2H$, $\psi'(x,z) = \frac{N^2h_0}{|m|f_0}\cos(kx)\exp\left(\frac{z}{2H} - |m|z\right)$

Thus, in the former case the wave propagates vertically and there is a lee-side trough, while in the latter the geopotential ridge coincides with the mountain ridge and the wave energy decays with height.

12.4. Consider a very simple model of a steady state mean meridional circulation for zonally symmetric flow in a midlatitude channel bounded by walls at $y = 0,\ \pi/l$ and $z = 0,\ \pi/m$. We assume that the zonal mean zonal flow $\overline{u}$ is in thermal wind balance and that the eddy momentum and heat fluxes vanish. For simplicity we let $\rho_0 = 1$ (Boussinesq approximation), and let the zonal force owing to small-scale motions be represented by a linear drag: $\overline{X} = -\gamma\overline{u}$. We assume that the diabatic heating has the form $J/c_p = (H/R)\,J_0 \cos ly \sin mz$, and we let N and f be constants. Eqs. (12.1), (12.2), (12.3), and (12.4) then yield the following:

$$-f_0\overline{v}^* = -\gamma\overline{u} \tag{i}$$

$$+N^2HR^{-1}\overline{w}^* = +\overline{J}/c_p \tag{ii}$$

$$\overline{v}^* = -\frac{\partial\overline{\chi}^*}{\partial z};\quad \overline{w}^* = \frac{\partial\overline{\chi}^*}{\partial y} \tag{iiia,b}$$

$$f_0\partial\overline{u}/\partial z + RH^{-1}\partial\overline{T}/\partial y = 0 \tag{iv}$$

Assuming that there is no flow through the walls, solve for the residual circulation defined by $\overline{\chi}^*$, $\overline{v}^*$, and $\overline{w}^*$.

Solution: From (ii), we immediately get $\overline{w}^* = (J_0/N^2)\cos(ly)\sin(mz)$. Then from (iii), $\overline{\chi}^* = (J_0N^{-2}l^{-1})\sin(ly)\sin(mz)$ and $\overline{v}^* = -(J_0/N^2)(m/l)\sin(ly)\cos(mz)$.

12.5. For the situation of problem 12.4 solve for the steady state zonal wind and temperature fields $\overline{u}$ and $\overline{T}$.

Solution: Now from equation (i) of problem 12.4 we find that $\overline{u} = \left(\frac{f_0}{\gamma}\right)\overline{v}^* = -\left(\frac{f_0}{\gamma}\right)\left(\frac{J_0}{N^2}\right)\left(\frac{m}{l}\right)\sin(ly)\cos(mz)$. And application of the thermal wind equation then gives $\overline{T} = \left(\frac{f_0^2}{\gamma}\right)\left(\frac{H}{R}\right)\left(\frac{J_0}{N^2}\right)\left(\frac{m}{l}\right)^2\cos(ly)\sin(mz)$.

12.6. Find the geopotential and vertical velocity fluctuations for a Kelvin wave of zonal wave number 1, phase speed 40 m s^{-1}, and zonal velocity perturbation amplitude 5 m s^{-1}. Let $N^2 = 4\times10^{-4}$ s^{-2}.

Solution: From (12.38) $\Phi' = u'(\nu/k) = u'c$, so $|\Phi'| = (40)(5) = 200\,\text{m}^2\,\text{s}^{-2}$ From (12.37) $w' \approx (\nu m/N^2)\Phi'$ since $|m| \gg 1/(2H)$. Also, $\nu = ck = cs/a$, where s is the planetary wavenumber. Now, $m^2 \approx (N/c)^2 = 4\times10^{-4}/1600 = 25\times10^{-8}$, so that $|m| = 5\times10^{-4}\,\text{m}^{-1}$. Substituting into the formula for vertical velocity:

$$|w'| = (40)\left(6.37\times10^6\right)^{-1}\left(5\times10^{-4}\right)(200)\left(4\times10^{-4}\right)^{-1} = 1.57\times10^{-3}\,\text{m}\,\text{s}^{-1}$$

12.7. For the situation of Problem 12.6 compute the vertical momentum flux $M \equiv \rho_0\overline{u'w'}$. Show that M is constant with height.

Solution: Now by definition: $\rho_0 = \rho_s\exp(-z^*/2H)$. Thus,
$M = \rho_s\exp(-z^*/H)\,\overline{u'w'}$. But $u' = U_0\cos(kx+mz-\nu t)\exp(+z^*/2H)$, and from Problem 12.6, $w' = (c^2km/N^2)$ $U_0\cos(kx+mz-\nu t)\exp(+z^*/2H)$. Thus, for $\rho_s = 1\,\text{kg}\,\text{m}^{-3}$

$$M = \rho_s\left(\frac{U_0^2c^2km}{N^2}\right)\overline{\cos^2(kx+mz-\nu t)} = \frac{(5)^2(40)^2\left(5\times10^{-4}\right)(0.5)}{\left(6.37\times10^6\right)\left(4\times10^{-4}\right)}$$

$$= 3.92\times10^{-3}\,\text{kg}\,\text{m}^{-1}\text{s}^{-1}$$

12.8. Determine the form for the vertical velocity perturbation for the Rossby-gravity wave corresponding to the u', v', and Φ' perturbations given in (12.44).

Solution: From (12.37) and (12.44), and with $|m| \gg 1/(2H)$:

$$\hat{w} \approx -\left(\nu m/N^2\right)\hat{\Phi} = -\hat{v}\left(\frac{i\nu^2 m}{N^2}\right)y\exp\left[-\frac{\beta|m|y^2}{2N}\right]$$

12.9. For a Rossby-gravity wave of zonal wave number 4 and phase speed –20 m s^{-1}, determine the latitude at which the vertical momentum flux $M \equiv \rho_0 \overline{u'w'}$ is a maximum.

Solution: From problem 12.8 and equation (12.44): $\overline{u'w'} = Ay^2 \exp\left[-\frac{\beta|m|y^2}{N}\right]$, where A is a constant. Then $\partial\overline{u'w'}/\partial y = 0 \rightarrow y^2\left(\beta\,|m|/N\right) = 1$, or $y_{\max} = \pm\nu\beta^{-1}\left(1 + k\nu/\beta\right)^{-1/2}$, where $\nu = ck = cs/a = (-20)\,(4)\,\left(6.37 \times 10^6\right)^{-1}\ \mathrm{s}^{-1}$. Thus, $y_{\max} = \pm\frac{cs}{2\Omega\left(1+0.5s\nu/\Omega\right)^{1/2}} = \frac{\pm 20\times 4}{\left(1.458\times 10^{-4}\right)(0.664)^{1/2}} = \pm 677.7\,\mathrm{km}$, or about 6° latitude north and south of the equator.

12.10. Suppose that the mean zonal wind shear in the descending westerlies of the equatorial QBO can be represented analytically on the equatorial β-plane in the form $\partial u/\partial z = \Lambda \exp\left(-y^2/L^2\right)$, where $L = 1200\,\mathrm{km}$. Determine the approximate meridional dependence of the corresponding temperature anomaly for $|y| \ll L$.

Solution: From (12.45) $\frac{\partial^2\overline{T}}{\partial y^2} = -\frac{\beta H}{R}\frac{\partial\overline{u}}{\partial z} = -\frac{\beta H\Lambda}{R}\exp\left(-\frac{y^2}{L^2}\right)$. An approximate solution can be obtained by expanding the y-dependence as follows:

$$\overline{T} = T_0 + T_1 \exp\left(-y^2/L^2\right) + T_2\left(y^2/L^2\right)\exp\left(-y^2/L^2\right) + \text{higher-order terms},$$

where T_0, T_1, and T_2 are constants. Substituting into the above equation and equating terms with the same y dependence gives $\frac{2}{L^2}\left(T_1 - T_2\right) = \frac{\beta H\Lambda}{R}$ and $-\frac{6}{L^4}T_2 + \frac{4}{L^4}T_1 - \frac{4}{L^4}T_2 = 0$. Thus, $T_2 = \frac{4}{10}T_1$ and $\mathrm{T}_1 = \left(\frac{5}{6}\right)\frac{\beta H\Lambda L^2}{R}\exp\left(\frac{-y^2}{L^2}\right)$.

12.11. Estimate the TEM residual vertical velocity in the westerly shear zone of the equatorial QBO assuming that radiative cooling can be approximated by Newtonian cooling with a 20-day relaxation time, the vertical shear is 20 m s^{-1} per 5 km, and the meridional half-width is 12° latitude.

Solution: In this case the temperature tendency term can be neglected, and (12.2) is an approximate balance between adiabatic warming and Newtonian cooling: $\left(N^2H/R\right)\overline{w}^* \approx \overline{J}/c_p = -\alpha\left(\overline{T} - T_0\right)$. But from Problem 12.10, near the equator we have $\overline{T} - T_0 \approx +\frac{\beta H\Lambda L^2}{R}\left(\frac{5}{6}\right)\exp\left(-\frac{y^2}{L^2}\right)$. Thus, $\overline{w}^* \approx -\frac{\alpha\beta\Lambda L^2}{N^2}\left(\frac{5}{6}\right)\exp\left(-\frac{y^2}{L^2}\right)$.
Now, $\alpha = 1/20$ days $= 5.8 \times 10^{-7}\ \mathrm{s}^{-1}$. So,

$$\overline{w}^* \approx -\frac{\left(5/6\right)\left(5.8\times 10^{-7}\right)\left(2.29\times 10^{-11}\right)\left(4\times 10^{-3}\right)\left(1.33\times 10^{6}\right)^2}{\left(4\times 10^{-4}\right)} = -1.96\times 10^{-4}\,\mathrm{m\,s^{-1}},$$

or $\overline{w}^* \approx -16.9\ \mathrm{m\,day^{-1}}$ at the equator. (Note that there is subsidence corresponding to the westerly shear phase of the QBO.)