Tropical Dynamics

11.1. Suppose that the relative vorticity at the top of an Ekman layer at 15°N is $\zeta = 2\times 10^{-5}\ \mathrm{s}^{-1}$. Let the eddy viscosity coefficient be $K_m = 10\ \mathrm{m}^2\,\mathrm{s}^{-1}$, and the water vapor mixing ratio at the top of the Ekman layer be $12\,\mathrm{g\,kg}^{-1}$. Use the method of Section 11.3 to estimate the precipitation rate owing to moisture convergence in the Ekman layer.

Solution: $P = \rho q w\,(De) = \rho q \zeta \left(\frac{K_m}{2f_0}\right)^{1/2} = (1.1)\left(12\times 10^{-3}\right)\left(2\times 10^{-5}\right)\left(\frac{10}{2\times 3.77\times 10^{-5}}\right)^{1/2} = 9.6\times 10^{-5}\,\mathrm{kg\,m}^{-2}\,\mathrm{s}^{-1}$. Dividing by water density $\rho_w = 10^3\,\mathrm{kg\,m}^{-3}$ gives $P/\rho_w = 9.6\times 10^{-8}\,\mathrm{m\,s}^{-1}$ or 8.3 mm/day.

11.2. As mentioned in Section 11.1.3, barotropic instability is a possible energy source for some equatorial disturbances. Consider the following profile for an easterly jet near the equator:

$$\overline{u}(y) = -u_0 \sin^2\left[l\,(y-y_0)\right]$$

where u_0, y_0, and l are constants and y is the distance from the equator. Determine the necessary conditions for this profile to be barotropically unstable.

Solution: $\beta - d^2\overline{u}/dy^2 < 0$ somewhere for instability. For this case:

$$d\overline{u}/dy = -2u_0 l \sin\left[l\,(y-y_0)\right]\cos\left[l\,(y-y_0)\right],\ \text{and}$$
$$d^2\overline{u}/dy^2 = 2u_0 l^2\left\{\sin^2\left[l\,(y-y_0)\right] - \cos^2\left[l\,(y-y_0)\right]\right\}.$$

Thus, $2u_0 l^2 > \beta$ for instability at $y = y_0$.

11.3. Show that the nonlinear terms in the balance equation (11.15)

$$G(x,y) \equiv -\nabla^2\left(\frac{1}{2}\nabla\psi\cdot\nabla\psi\right) + \nabla\cdot\left(\nabla\psi\nabla^2\psi\right)$$

may be written in Cartesian coordinates as

$$G(x,y) = 2\left[\left(\partial^2\psi/\partial x^2\right)\left(\partial^2\psi/\partial y^2\right) - \left(\partial^2\psi/\partial x\partial y\right)^2\right].$$

Solution: Let $A = -\nabla^2\left(\frac{\nabla\psi\cdot\nabla\psi}{2}\right) = -\frac{1}{2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left[\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2\right]$

$$= -\frac{1}{2}\left\{\frac{\partial}{\partial x}\left[2\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial x^2} + 2\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y}\right] + \frac{\partial}{\partial y}\left[2\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial x\partial y} + 2\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial y^2}\right]\right\}$$

$$= -\frac{\partial^2\psi}{\partial x^2}\frac{\partial^2\psi}{\partial x^2} - \frac{\partial\psi}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial^2\psi}{\partial x^2}\right) - \left(\frac{\partial^2\psi}{\partial x\partial y}\right)^2 - \frac{\partial\psi}{\partial y}\frac{\partial^3\psi}{\partial x^2\partial y}$$

$$-\left(\frac{\partial^2\psi}{\partial x\partial y}\right)^2 - \frac{\partial\psi}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial^2\psi}{\partial y^2}\right) - \left(\frac{\partial^2\psi}{\partial y^2}\right)^2 - \frac{\partial\psi}{\partial y}\frac{\partial}{\partial y}\left(\frac{\partial^2\psi}{\partial y^2}\right)$$

$$B = \nabla\cdot\left(\nabla\psi\nabla^2\psi\right) = \frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial x}\nabla^2\psi\right) + \frac{\partial}{\partial y}\left(\frac{\partial\psi}{\partial y}\nabla^2\psi\right)$$

$$= \frac{\partial^2\psi}{\partial x^2}\nabla^2\psi + \frac{\partial\psi}{\partial x}\frac{\partial}{\partial x}\nabla^2\psi + \frac{\partial^2\psi}{\partial y^2}\nabla^2\psi + \frac{\partial\psi}{\partial y}\frac{\partial}{\partial y}\nabla^2\psi.$$

The sum of $A + B = 2\left[\left(\partial^2\psi/\partial x^2\right)\left(\partial^2\psi/\partial y^2\right) - \left(\partial^2\psi/\partial x\partial y\right)^2\right]$.

11.4. With the aid of the results of Problem 11.3, show that if f is assumed to be constant, the balance equation (11.15) is equivalent to the gradient wind equation (3.15) for a circularly symmetric regular low with geopotential perturbation given by $\Phi = \Phi_0\left(x^2 + y^2\right)/L^2$, where Φ_0 is a constant geopotential and L a constant length scale. Hint: Assume that $\psi(x, y)$ has the same functional dependence on (x, y) as does Φ.

Solution: For cylindrical coordinates (3.15) becomes for a regular low

$V = \frac{\partial\psi}{\partial r} = -\frac{fr}{2} \pm \left[\left(\frac{fr}{2}\right)^2 + r\frac{\partial\Phi}{\partial r}\right]^{1/2}$ where $r = \left(x^2 + y^2\right)^{1/2}$ is the distance from the center of the vortex. Thus, $\Phi = \Phi_0\left(r^2/L^2\right)$, and $\partial\Phi/\partial r = 2r\Phi_0/L^2$. Thus, from (3.15) if ψ has the same dependence on r as Φ:

(I) $\frac{\partial\psi}{\partial r} = \frac{2r\psi_0}{L^2} = -\frac{fr}{2} \pm \left[\left(\frac{fr}{2}\right)^2 + \frac{2r^2\Phi_0}{L^2}\right]^{1/2}$.

But from (11.15) $\nabla^2\Phi = f\nabla^2\psi + 2\left[\left(\partial^2\psi/\partial x^2\right)\left(\partial^2\psi/\partial y^2\right) - \left(\partial^2\psi/\partial x\partial y\right)^2\right]$, which after substituting the x and y dependencies of Φ and ψ gives $\Phi_0 = f\psi_0 + 2\psi_0^2/L^2$. Solving for ψ_0 gives:

(II) $\psi_0 = -\frac{fL^2}{4} \pm \frac{1}{2}\left[\left(\frac{fL^2}{2}\right)^2 + 2L^2\Phi_0\right]^{1/2}$. Comparing (I) and (II), we see that they differ by the common factor $2r/L^2$, so the two expressions are proportional, as was to be shown.

11.5. Starting from the perturbation equations (11.27), (11.28), and (11.29), show that the sum of kinetic plus available potential energy is conserved for equatorial waves. Hence, show that for the Kelvin wave there is an equipartition of energy between kinetic and available potential energy.

Solution: Take $u' \times$ (11.27) + $v' \times$ (11.28) + $\Phi'/(gh_e) \times$ (11.29). The result is $\frac{1}{2}\frac{\partial}{\partial t}\left(u'^2 + v'^2 + \frac{\Phi'^2}{gh_e}\right) + \frac{\partial}{\partial x}\left(u'\Phi'\right) + \frac{\partial}{\partial y}\left(v'\Phi'\right) = 0$. Averaging over a wavelength in x and for $-\infty < y < \infty$, and using the boundary conditions that perturbations vanish for $|y| \to \infty$ yields $\int_{-\infty}^{+\infty}\int_0^{L_x}\frac{1}{2}\frac{\partial}{\partial t}\left(u'^2 + v'^2 + \frac{\Phi'^2}{gh_e}\right)dxdy = 0$. Thus, the area average $\left\langle u'^2 + v'^2 + \frac{\Phi'^2}{gh_e}\right\rangle =$ Constant. For Kelvin waves the meridional velocity vanishes and from (11.43) and (11.46) $\Phi' = \sqrt{gh_e}u'$, which immediately shows that $\left\langle u'^2\right\rangle = \left\langle \Phi'^2/(gh_e)\right\rangle$, so that there is equipartition of energy.

11.6. Solve for the meridional dependence of the zonal wind and geopotential perturbations for a Rossby-gravity mode in terms of the meridional velocity distribution (11.38).

Solution: From (11.35),

$$\hat{\Phi} = \frac{-i\nu gh_e}{\nu^2 - k^2gh_e}\left(\frac{\partial\hat{v}}{\partial y} - \frac{k\beta}{\nu}y\hat{v}\right) = \frac{i\nu gh_e}{\nu^2 - k^2gh_e}\left(\frac{\beta}{\sqrt{gh_e}} + \frac{k\beta}{\nu}\right)y\hat{v} = \frac{i\beta y\hat{v}}{\left(\nu/\sqrt{gh_e} - k\right)}$$

But from (11.39) $\nu/\sqrt{gh_e} - k = \beta/\nu$. Thus, $\hat{\Phi}(y) = i\nu y\hat{v}(y)$, and from (11.31) $-i\nu\hat{u} = \beta y\hat{v} - ik\hat{\Phi} = (\beta + k\nu)y\hat{v}$, so that $\hat{u}(y) = i\nu^{-1}(\beta + k\nu)y\hat{v}$.

11.7. Use the linearized model (11.48) and (11.49) to compute the meridional distribution of divergence in the mixed layer for a situation in which the geostrophic wind is given by $u_g = u_0 \exp\left(-\beta y^2/2c\right)$, $v_g = 0$, where u_0 and c are constants.

Solution: From (11.48−49) $\alpha u - \beta y v = 0$; $\alpha v + \beta y u = \beta y u_g$. Thus, $u = \beta y v/\alpha$ and $v = \alpha\beta y u_g/\left(\alpha^2 + \beta^2 y^2\right)$. So, $\frac{\partial v}{\partial y} = \alpha\beta\left[\frac{(u_g + y\partial u_g/\partial y)}{\alpha^2+\beta^2y^2} - \frac{2\beta^2y^2u_g}{(\alpha^2+\beta^2y^2)^2}\right]$, or $\frac{\partial v}{\partial y} = \frac{\alpha\beta u_0}{(\alpha^2+\beta^2y^2)^2}\left[\alpha^2\left(1-\frac{\beta y^2}{c}\right) - \beta^2y^2 - \frac{\beta^3y^4}{c}\right]\exp\left(-\beta y^2/2c\right)$.

11.8. Show that the frequency of the $n = 1$ equatorial Rossby mode is given approximately by $\nu = -k\beta\left(k^2 + 3\beta/\sqrt{gh_e}\right)^{-1}$, and use this result to solve for the $\hat{u}(y)$ and $\hat{\Phi}(y)$ fields in terms of $\hat{v}(y)$. Hint: Use the fact that the Rossby wave phase speed is much less than $\sqrt{gh_e}$.

Solution: From (11.37), if $(\nu/k)^2 \ll gh_e$, then $\frac{\sqrt{gh_e}}{\beta}\left(-k^2 - \frac{k\beta}{\nu}\right) \approx 3$, which can be solved for ν to give $\nu = -k\beta\left(k^2 + 3\beta/\sqrt{gh_e}\right)^{-1}$. From (11.40) the meridional velocity can be expressed as

$$\hat{v}(y) = 2v_0\left(\frac{\beta}{\sqrt{gh_e}}\right)^{1/2} y \exp\left[-\left(\frac{\beta}{\sqrt{gh_e}}\right)\frac{y^2}{2}\right].$$

Solving (11.35) for $\hat{\Phi}(y)$, while again using the fact that $\nu^2/k^2 \ll gh_e$, yields $\hat{\Phi}(y) = \frac{i\nu}{k^2}\left(\frac{\partial\hat{v}}{\partial y} - \frac{k}{\nu}\beta y\hat{v}\right)$. Substituting from the expression for $\hat{v}(y)$ then yields $\hat{\Phi}(y) = 2i v_0\left(\frac{\beta}{\sqrt{gh_e}}\right)^{1/2}\left(\frac{\nu}{k^2}\right)\left[1 - \beta\left(\frac{k}{\nu} + \frac{1}{\sqrt{gh_e}}\right)y^2\right]\exp\left[-\left(\frac{\beta}{\sqrt{gh_e}}\right)\frac{y^2}{2}\right]$. Solving (11.31) for $\hat{u}(y)$ gives $\hat{u}(y) = i(\beta/\nu)y\hat{v} + (k/\nu)\hat{\Phi}$, so that $\hat{u}(y) = 2iv_0\left(\frac{\beta}{\sqrt{gh_e}}\right)^{1/2}\left(\frac{1}{k}\right)\left[1 - \frac{\beta}{\sqrt{gh_e}}y^2\right]\exp\left[-\left(\frac{\beta}{\sqrt{gh_e}}\right)\frac{y^2}{2}\right]$. (Note that the zonal wind changes sign at $y^2 = \sqrt{gh_e}/\beta$.)