Chapter 10

The General Circulation

10.1. Starting with the isobaric version of the thermodynamic energy equation (2.42), derive the log-pressure version (10.5).

Solution: Using the ideal gas law (2.42) can be rewritten as:
(I) $\frac{DT}{Dt} - \frac{RT}{c_p p}\omega = \frac{J}{c_p}$, where $\omega \equiv Dp/Dt$. But,
(II) $\frac{\omega}{p} = \frac{1}{p}\frac{Dp}{Dt} = \frac{D\ln p}{Dt} = -\frac{w^*}{H}$, where $w^* \equiv Dz^*/Dt = -HD(\ln p)/Dt$.
Substituting from (II) into (I) and noting that $\kappa \equiv R/c_p$ gives $\frac{DT}{Dt} + \frac{\kappa T}{H}w^* = \frac{J}{c_p}$.

10.2. Show that in the σ-coordinate system a mass element $\rho_0 dxdydz$ takes the form $-g^{-1}p_s dxdyd\sigma$.

Solution: By definition $\sigma \equiv p/p_s$. But $p_s = p_s(x, y, t)$. So, with the aid of the hydrostatic equation: $\frac{\partial \sigma}{\partial z} = \frac{1}{p_s}\frac{\partial p}{\partial z} = -\frac{\rho_0 g}{p_s}$. Thus, $p_s d\sigma = -\rho_0 g dz$, from which $\rho_0 dxdydz = -p_s g^{-1} dxdyd\sigma$.

10.3. Compute the mean zonal wind $\overline{u}$ at the 200 hPa level at 30°N under the assumptions that $\overline{u} = 0$ at the equator, and that the absolute angular momentum is independent of latitude. What is the implication of this result for the role of eddy motions?

Solution: By conservation of angular momentum: $\Omega R_0^2 = (\Omega + u/R_1)R_1^2$. But if R_0 is the distance from the axis of rotation at 0° and R_1 at 30°, then $R_1 = R_0\left(\sqrt{3}/2\right)$, and hence, $u = \frac{\Omega\left(R_0^2 - R_1^2\right)}{R_1} = \frac{\Omega R_0}{2\sqrt{3}} = 134\,\mathrm{m\,s^{-1}}$. This is far greater than the observed velocity, and indicates that there must be an eddy momentum flux divergence between the equator and 30° latitudes to balance the tendency of the mean Hadley cell to produce a constant angular momentum profile in the upper troposphere.

10.4. Show by scale analysis that advection by the mean meridional circulation can be neglected in the zonally averaged equations (10.11) and (10.12) for quasi-geostrophic motions.

Solution: To estimate terms in (10.11) let scales be $|\overline{u}| \sim U \sim 10\,\mathrm{m\,s^{-1}}$, time scale be 10^5 s, then $\partial\overline{u}/\partial t \sim 10^{-4}\,\mathrm{m\,s^{-2}}$, and from Figure 10.6 note that $\left|\partial\left(\overline{u'v'}\right)/\partial y\right| \leq 10^{-4}\,\mathrm{m\,s^{-2}}$ Now $|\overline{v}\partial\overline{u}/\partial y|$ and $|\overline{w}\partial\overline{u}/\partial z| < |f_0\overline{v}|$ for quasi-geostrophic flow. Thus, for balance $f_0\overline{v} \leq 10^{-4}$, and hence $|\overline{v}| \leq 1\,\mathrm{m\,s^{-1}}$. Letting the meridional scale be $L \sim 10^6$ m, and the vertical scale be $H \sim 10^4$ m, we then see that $\overline{w} \sim (H/L)\,\overline{v} \sim 10^{-2}\,\mathrm{m\,s^{-1}}$. Thus, $|\overline{v}\partial\overline{u}/\partial y| \sim |\overline{w}\partial\overline{u}/\partial z| \sim 10^{-5}\,\mathrm{m\,s^{-2}}$. Hence, the leading terms are: $\frac{\partial\overline{u}}{\partial t} - f_0\overline{v} = -\frac{\partial\left(\overline{u'v'}\right)}{\partial y}$.
For the thermodynamic equation (10.12), we first observe that from Figure 10.3 that $\left|\partial\left(\overline{v'T'}\right)/\partial y\right| \sim 10^{-5}\,\mathrm{K\,s^{-1}}$. The hypsometric equation shows that $\overline{T} \sim \overline{\Phi}/R$, while from geostrophy $\overline{\Phi} \sim f_0 UL \sim 10^3\,\mathrm{m^2\,s^2}$, hence, $\overline{T} \sim 3\,\mathrm{K}$. Thus, $\partial\overline{T}/\partial t \sim 3 \times 10^{-5}\,\mathrm{K\,s^{-1}}$, and $\overline{w}\left(N^2 H/R\right) \sim 10^{-2}\left(10^{-4} \times 10^4/287\right) \sim 3 \times 10^{-5}\,\mathrm{K\,s^{-1}}$. But, $\overline{v}\partial\overline{T}/\partial y \sim \overline{w}\partial\overline{T}/\partial z \sim 3 \times 10^{-6}\,\mathrm{K\,s^{-1}}$. So these two advection terms can be neglected in a first approximation.

10.5. Show that for quasi-geostrophic eddies the next to last term in square brackets on the right-hand side in (10.15) is proportional to the vertical derivative of the eddy meridional relative vorticity flux.

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Solution: From geostrophy: $f_0 u' = -\partial\Phi'/\partial y$ and $f_0 v' = \partial\Phi'/\partial x$. Thus, $\partial u'/\partial x + \partial v'/\partial y = 0$. So, $-\frac{\partial\left(\overline{u'v'}\right)}{\partial y} = -\overline{u'\frac{\partial v'}{\partial y}} - \overline{v'\frac{\partial u'}{\partial y}} = \overline{u'\frac{\partial u'}{\partial x}} - \overline{v'\frac{\partial u'}{\partial y}} = -\overline{v'\frac{\partial u'}{\partial y}}$, where we have used the fact that from cyclic continuity in x: $\overline{u'\frac{\partial u'}{\partial x}} = \frac{\partial\overline{(u'^2/2)}}{\partial x} = 0$. And, $\overline{v'\zeta'} = \overline{v'\frac{\partial v'}{\partial x}} - \overline{v'\frac{\partial u'}{\partial y}} = -\overline{v'\frac{\partial u'}{\partial y}}$ since $\overline{v'\frac{\partial v'}{\partial x}} = \frac{\partial\overline{(v'^2/2)}}{\partial x} = 0$.

Thus, $-\frac{\partial\left(\overline{u'v'}\right)}{\partial y} = \overline{v'\zeta'}$, or $\frac{\partial\left(\overline{v'\zeta'}\right)}{\partial z} = -\frac{\partial}{\partial z}\left[\frac{\partial\left(\overline{u'v'}\right)}{\partial y}\right]$

10.6. Starting from equations (10.16)–(10.19), derive the governing equation for the residual streamfunction (10.21).

Solution: Taking $f_0\frac{\partial(10.17)}{\partial z} + \frac{R}{H}\frac{\partial(10.18)}{\partial y}$ and applying the thermal wind relationship: $f_0\partial\overline{u}/\partial z = -RH^{-1}\partial\overline{T}/\partial y$ gives

$$-f_0^2\frac{\partial\overline{v}^*}{\partial z} + N^2\frac{\partial\overline{w}^*}{\partial y} = f_0\frac{\partial\overline{G}}{\partial z} + \frac{\kappa}{H}\frac{\partial\overline{J}}{\partial y} \tag{A}$$

But, by definition $\rho_0\overline{v}^* = -\partial\overline{\chi}^*/\partial z$ and $\rho_0\overline{w}^* = \partial\overline{\chi}^*/\partial y$ so substituting into (A) and multiplying be ρ_0 gives: $\frac{\partial^2\overline{\chi}^*}{\partial y^2} + \frac{\rho_0 f_0^2}{N^2}\frac{\partial}{\partial z}\left(\frac{1}{\rho_0}\frac{\partial\overline{\chi}^*}{\partial z}\right) = \frac{\rho_0}{N^2}\left[f_0\frac{\partial\overline{G}}{\partial z} + \frac{\kappa}{H}\frac{\partial\overline{J}}{\partial y}\right]$.

10.7. Using the observed data given in Figure 10.13 of the text, compute the time required for each possible energy transformation or loss to restore or deplete the observed energy stores. (A watt equals 1 J s^{-1}.)

Solution: Dividing the energy stores by each possible transformation and source/sink gives replacement/depletion times as shown in the diagram (in seconds).

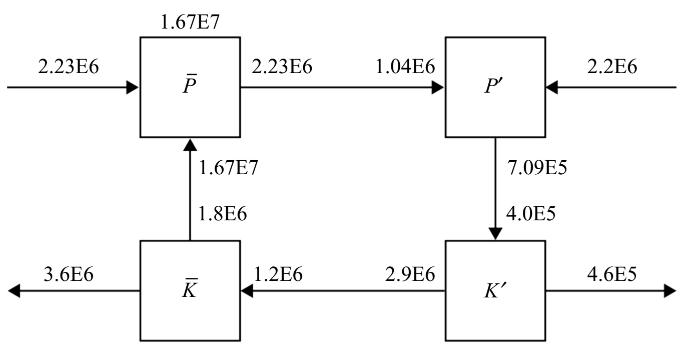

FIGURE 10.7

10.8. Compute the surface torque per unit horizontal area exerted on the atmosphere by topography for the following distribution of surface pressure and surface height:

$$p_s = p_0 + \hat{p}\sin kx, \qquad h = \hat{h}\sin(kx - \gamma)$$

where $p_0 = 1000$ hPa, $\hat{p} = 10$ hPa, $\hat{h} = 2.5 \times 10^3$ m, $\gamma = \pi/6$ rad, and $k = 1/(a\cos\phi)$. Here, $\phi = \pi/4$ radians is the latitude, and a is the radius of the earth. Express the answer in kg s^{-2}.

Solution: Using angle brackets to designate the longitudinal average, from (10.43): Torque $= -(a\cos\phi)\langle p_s\partial h/\partial x\rangle = -(a\cos\phi)\left\langle\left(p_0 + \hat{p}\sin kx\right)\left(\hat{h}k\cos(kx-\gamma)\right)\right\rangle$

Torque $= -(a\cos\phi)\hat{p}\hat{h}k\left\langle\sin kx\cos kx\cos\gamma + \sin^2 kx\sin\gamma\right\rangle = -(\hat{p}\hat{h}/2)\sin\gamma = -(10^3)\left(2.5\times10^3/2\right)\sin(\pi/6) = -6.25\times10^5\ \mathrm{kg\,s^{-2}}$.

10.9. Starting from (10.66) and (10.67) show that the group velocity relative to the ground for stationary Rossby waves is perpendicular to the wave crests and has a magnitude given by (10.69).

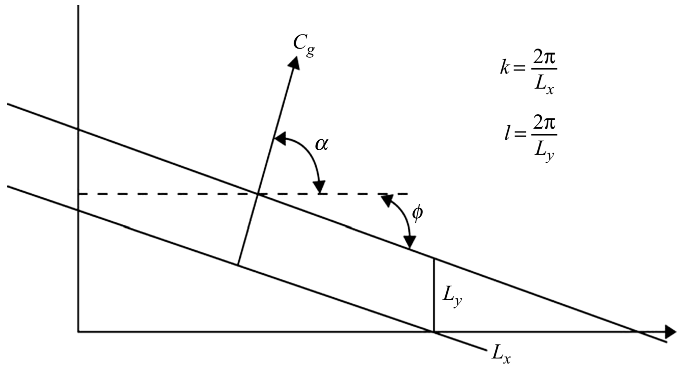

FIGURE 10.9

Solution: For stationary waves (10.65) shows that $\overline{u} = \beta/\left(k^2 + l^2\right)$. Substituting this expression into (10.66) and (10.67) we get $c_{gy} = 2\overline{u}kl/\left(k^2 + l^2\right)$ and $c_{gx} = \overline{u} + \overline{u}\left(k^2 - l^2\right)/\left(k^2 + l^2\right) = 2\overline{u}k^2/\left(k^2 + l^2\right)$. Then $c_{gy}/c_{gx} = \tan\alpha = l/k$, and $\tan\phi = L_y/L_x = k/l = \cot(\pi/2 - \phi) = \cot\alpha$. Thus, $\phi + \alpha = 90^\circ$ as was to be shown.

10.10. Consider a thermally stratified liquid contained in a rotating annulus of inner radius 0.8 m, outer radius 1.0 m, and depth 0.1 m. The temperature at the bottom boundary is held constant at T_0. The fluid is assumed to satisfy the equation of state (10.75) with $\rho_0 = 10^3$ kg m^{-3} and $\varepsilon = 2 \times 10^{-4}$ K^{-1}. If the temperature increases linearly with height along the outer radial boundary at a rate of 1°C cm^{-1} and is constant with height along the inner radial boundary, determine the geostrophic velocity at the upper boundary for a rotation rate of $\Omega = 1$ rad s^{-1}. (Assume that the temperature depends linearly on radius at each level.)

Solution: From equation (10.76) $\frac{\partial u_g}{\partial z} = \left(\frac{\varepsilon g}{2\Omega}\right)\frac{\partial T}{\partial r}$ where the r-coordinate is taken to be the distance from the inner wall. Then $T(r, z) = T_0 + (r/L)\,[z\,(dT/dz)]$, with dT/dz the temperature gradient along the outer wall. Thus, $\frac{\partial u_g}{\partial z} = \left(\frac{\varepsilon g}{2\Omega}\right)\left(\frac{dT}{dz}\right)\left(\frac{z}{L}\right)$, and

$$u_g = \left(\frac{\varepsilon g}{2\Omega L}\right)\left(\frac{dT}{dz}\right)\int_0^H z\,dz = \frac{\left(2 \times 10^{-4}\right)(9.8)}{(2)\,(0.2)}\,(100)\,\frac{(0.1)^2}{2} = 0.00245\ \text{m s}^{-1}.$$

10.11. Show by considering $\partial\overline{u}/\partial t$ for small perturbations about the equilibrium points in Figure 10.18 that point B is an unstable equilibrium point while points A and C are stable.

Solution: $\partial\overline{u}/\partial t = -D(\overline{u}) - \kappa(\overline{u} - U)$, so for equilibrium $D(\overline{u}) = -\kappa(\overline{u} - U)$. From Figure 10.17 it is clear that for points A and C the following hold:

If $\delta\overline{u} > 0$ then $D(\overline{u}) > \kappa(\overline{u} - U)$ so $\partial\overline{u}/\partial t < 0$
If $\delta\overline{u} < 0$ then $D(\overline{u}) < \kappa(\overline{u} - U)$ so $\partial\overline{u}/\partial t > 0$

In both cases, $\partial\overline{u}/\partial t$ is opposite in sign to $\delta\overline{u}$ so the perturbation is damped and equilibrium is *stable*.
For Point B:

If $\delta\overline{u} > 0$ then $D(\overline{u}) < \kappa(\overline{u} - U)$ so $\partial\overline{u}/\partial t > 0$
If $\delta\overline{u} < 0$ then $D(\overline{u}) > \kappa(\overline{u} - U)$ so $\partial\overline{u}/\partial t < 0$

In both cases, $\partial\overline{u}/\partial t$ is same sign as $\delta\overline{u}$ so the perturbation amplifies and equilibrium is *unstable*.