Chapter 9

Mesoscale Circulations

9.1. Show by transforming from θ-coordinates to height coordinates that the Ertel potential vorticity P is proportional to $F^2N_s^2 - S^4$. See equation (9.28).

Solution: $\overline{P} = -g\left[f - \left(\partial\overline{u}_g/\partial y\right)_\theta\right](\partial\theta/\partial p)$ in θ-coordinates. But, $-\left(\partial\overline{u}_g/\partial y\right)_\theta = -\left(\partial\overline{u}_g/\partial y\right)_z + \left(\partial\overline{u}_g/\partial\theta\right)(\partial\theta/\partial y)_z$, and with the aid of (9.10) $\left(\partial\overline{u}_g/\partial\theta\right) = \left(\partial\overline{u}_g/\partial z\right)(\partial\theta/\partial z)^{-1} = -(g/f\theta_0)(\partial\theta/\partial y)(\partial\theta/\partial z)^{-1}$. Also, $-g(\partial\theta/\partial p) = \rho^{-1}(\partial\theta/\partial z)$. Thus, substituting into the top expression gives $\overline{P} = \frac{1}{\rho}\left[f - \left(\frac{\partial u_g}{\partial y}\right)_z\right]\left(\frac{\partial\theta}{\partial z}\right) - \frac{g}{\rho f\theta_0}\left(\frac{\partial\theta}{\partial y}\right)^2$.

Hence, $\frac{\rho f g}{\theta_0}\overline{P} = f\left(f - \frac{\partial u_g}{\partial y}\right)\left(\frac{g}{\theta_0}\frac{\partial\theta}{\partial z}\right) - \left(\frac{g}{\theta_0}\frac{\partial\theta}{\partial y}\right)^2 = F^2N_s^2 - S^4$.

9.2 Starting with the linearized Boussinesq equations for a basic state zonal flow that is a function of height, derive (9.35) and verify the form given for the Scorer parameter.

Solution: For steady waves in mean flow $\overline{u}(z)$, eqs. (5.59)–(5.62) become

$$\overline{u}\frac{\partial u'}{\partial x} + w'\frac{\partial\overline{u}}{\partial z} + \frac{1}{\rho_0}\frac{\partial p'}{\partial x} = 0 \tag{1}$$

$$\overline{u}\frac{\partial w'}{\partial x} + \frac{1}{\rho_0}\frac{\partial p'}{\partial z} - \frac{\theta'}{\overline{\theta}}g = 0 \tag{2}$$

$$\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0 \tag{3}$$

$$\overline{u}\frac{\partial\theta'}{\partial x} + w'\frac{d\overline{\theta}}{dz} = 0. \tag{4}$$

Taking $\partial(2)/\partial x - \partial(1)/\partial z$ yields

$$\overline{u}\frac{\partial}{\partial x}\left(\frac{\partial w'}{\partial x} - \frac{\partial u'}{\partial z}\right) - \frac{\partial\overline{u}}{\partial z}\left(\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z}\right) - w'\frac{\partial^2\overline{u}}{\partial z^2} - \frac{g}{\overline{\theta}}\frac{\partial\theta'}{\partial x} = 0. \tag{5}$$

Using (3) and (4), we can eliminate u' and θ' in (5) to get $\frac{\partial^2 w'}{\partial x^2} + \frac{\partial^2 w'}{\partial z^2} + \left(\frac{1}{\overline{u}^2}\frac{g}{\overline{\theta}}\frac{d\overline{\theta}}{dz} - \frac{1}{\overline{u}}\frac{\partial^2\overline{u}}{\partial z^2}\right)w' = 0$, as was to be shown.

9.3. Show that for stationary flow over an isolated ridge in the broad ridge limit ($k_s \ll m_s$), the group velocity vector is directed upward so that energy cannot propagate upstream or downstream of the ridge.

Solution: In the wide ridge limit $m = -N/\overline{u}$, where the minus sign is required to make the group velocity upward. Then substituting into the expression for the horizontal group velocity (7.45a) gives $c_{gx} = \overline{u} + N/m = \overline{u} + N/(-N/\overline{u}) = 0$.

9.4. An air parcel at 920 hPa with temperature 20°C is saturated (mixing ratio 16 g kg^{-1}). Compute θ_e for the parcel.

Solution: $\theta_e = \theta\exp\left[(L_cq_s)/(c_pT)\right]$, where $\theta = T(p_s/p)^{R/c_p}$. Then, $\ln\theta_e = \ln T + \frac{R}{c_p}\ln\left(\frac{p_s}{p}\right) + \left(\frac{L_cq_s}{c_pT}\right) = \ln(293.15) + \frac{2}{7}\ln\left(\frac{1000}{920}\right) + \left[\frac{2.5\times10^6(16\times10^{-3})}{(1004)(293.15)}\right]$ so that $\ln\theta_e = 5.84$, and $\theta_e = 344$ K.

9.5. Suppose that the mass of air in an entraining cumulus updraft increases exponentially with height so that $m = m_0 e^{z/H}$, where $H = 8$ km and m_0 is the mass at a reference level. If the updraft speed is $3\ \text{m s}^{-1}$ at 2 km height, what is its value at a height of 8 km assuming that the updraft has zero net buoyancy?

Solution: From eq. (9.52) a neutrally buoyant updraft has $T_{cld} = T_{env}$ so that $\frac{d}{dz}\left[\ln\left(w'^2\right)\right] = -\left(2\frac{d\ln m}{dz}\right)$, which for $m = m_0 \exp(z/H)$ with $H = 8$ km gives $\ln\left(w^2\right)\Big|_{z=2}^{z=8} = -\left(\frac{2}{H}\right)(8-2) = -\frac{12}{8}$.
$w = 3\ \text{m s}^{-1}$ at $z = 2$ km. Thus, $w = 1.42\ \text{m s}^{-1}$ at 8 km.

9.6. Verify the approximate relationship between moist static energy and θ_e given by (9.41).

Solution: Forming the differential of the second line of the solution to 9.5 (assuming that L_c is constant) and multiplying through by c_pT yields $c_pTd\ln\theta_e = c_pdT - (RT/p)\,dp + L_cdq_s - L_cq_sd\ln T$. Substituting from the ideal gas law and the hydrostatic relation gives $(-RT/p)\,dp = gdz$. But $\frac{L_cdq_s}{L_cq_sd\ln T} = \frac{d\ln q_s}{d\ln T}$, and $\left|\frac{d\ln q_s}{d\ln T}\right| \gg 1$ (as can be verified from a thermodynamic diagram). Thus, $c_pTd\ln\theta_e \approx c_pdT + gdz + L_cdq_s = dh$.

9.7. The azimuthal velocity component in some hurricanes is observed to have a radial dependence given by $v_\lambda = V_0(r_0/r)^2$ for distances from the center given by $r \geq r_0$. Letting $V_0 = 50\ \text{m s}^{-1}$ and $r_0 = 50$ km, find the total geopotential difference between the far field ($r \to \infty$) and $r = r_0$, assuming gradient wind balance and $f_0 = 5 \times 10^{-5}\ \text{s}^{-1}$. At what distance from the center does the Coriolis force equal the centrifugal force?

Solution: From eq. (9.61) $\frac{V_0^2 r_0^4}{r^5} + \frac{f\,V_0 r_0^2}{r^2} = \frac{\partial\Phi}{\partial r}$. Integrating in r gives: $\int_{r_0}^{\infty} d\Phi = r_0^4 V_0^2 \int_{r_0}^{\infty} r^{-5}dr + f\,V_0 r_0^2 \int_{r_0}^{\infty} r^{-2}dr$. Thus, $\Phi(\infty) - \Phi(r_0) = V_0^2/4 + fV_0r_0$, or $\Phi(\infty) - \Phi(r_0) = 50^2/4 + \left(5 \times 10^{-5}\right)(50)\left(5 \times 10^4\right) = 750\ \text{m}^2\text{s}^{-2}$. The two terms on the left in (9.61) are equal when $\frac{V_0^2 r_0^4}{r^5} = \frac{f\,V_0 r_0^2}{r^2}$, or $r^3 = \frac{V_0 r_0^2}{f}$, so $r = 135.7$ km.

9.8. Starting with (9.61) derive the angular momentum form of the gradient wind balance for an axisymmetric vortex given by (9.62).

Solution: By definition $M_\lambda = v_\lambda r + fr^2/2$, so $v_\lambda = -fr/2 + M_\lambda/r$. Thus, $\frac{v_\lambda^2}{r} = \frac{f^2 r}{4} - \frac{fM_\lambda}{r} + \frac{M_\lambda^2}{r^3}$, and $f v_\lambda = -\frac{f^2 r}{2} + \frac{fM_\lambda}{r}$.
Thus, the gradient wind can be expressed as $\left(\frac{v_\lambda^2}{r} + f v_\lambda\right) = \frac{M_\lambda^2}{r^3} - \frac{f^2 r}{4} = \frac{\partial\Phi}{\partial r}$.