Chapter 8

The Planetary Boundary Layer

8.1. Verify by direct substitution that the Ekman spiral expression (8.31) is indeed a solution of the boundary layer equations (8.26) and (8.27) for the case $v_g = 0$.

Solution: $u = u_g(1 - e^{-\gamma z}\cos\gamma z)$; $v = u_g e^{-\gamma z}\sin\gamma z$. Thus, $\frac{\partial^2 u}{\partial z^2} = -2u_g\gamma^2 e^{-\gamma z}\sin\gamma z$; $\frac{\partial^2 v}{\partial z^2} = -2u_g\gamma^2 e^{-\gamma z}\cos\gamma z$, and substituting into (8.26) and (8.27) gives $-2\gamma^2 K u_g + f u_g = 0$. Hence, $\gamma^2 = f/2K$, so solution checks.

8.2. Derive the Ekman spiral solution for the more general case where the geostrophic wind has both x and y components (u_g and v_g, respectively), which are independent of height.

Solution: In this case the general solution of (8.20) is $u + iv = A\exp\left[(if/K)^{1/2}z\right] + B\exp\left[-(if/K)^{1/2}z\right] + \left(u_g + iv_g\right)$. Applying the boundary conditions (8.28) for $f > 0$ gives $A = 0$ and $B = -(u_g + iv_g)$. Taking the real and imaginary parts yields the velocity components in the Ekman layer:

$$u = u_g(1 - e^{-\gamma z}\cos\gamma z) - v_g e^{-\gamma z}\sin\gamma z$$
$$v = v_g(1 - e^{-\gamma z}\cos\gamma z) + u_g e^{-\gamma z}\sin\gamma z.$$

8.3. Letting the Coriolis parameter and density be constants, show that (8.38) is correct for the more general Ekman spiral solution obtained in Problem 8.2.

Solution: For ρ constant, from (8.36): $w(De) = -\left(\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y}\right)\int_0^{De}\left(1 - e^{-\gamma z}\cos\gamma z\right)dz + \left(\frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y}\right)\int_0^{De} e^{-\gamma z}\sin\gamma z dz$, But $\left(\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y}\right) = 0$; $\zeta_g = \left(\frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y}\right)$, $\therefore w(De) = \zeta_g(K/2f)^{1/2}$ (Northern Hemisphere).

8.4. For laminar flow in a rotating cylindrical vessel filled with water (molecular kinematic viscosity $\nu = 0.01\,\mathrm{cm^2\,s^{-1}}$), compute the depth of the Ekman layer and the spin-down time if the depth of the fluid is 30 cm and the rotation rate of the tank is ten revolutions per minute. How small would the radius of the tank have to be in order that the time scale for viscous diffusion from the side walls to be comparable to the spin-down time?

Solution: $De = \pi/\gamma$, where $\gamma = (\Omega/\nu)^{1/2}$. But $\Omega = 2\pi/6\,\mathrm{s^{-1}}$, so $De = 0.307$ cm. Spin-down time is given by $\tau = H(1/\Omega\nu)^{1/2} = 293$ s. The diffusive time scale is L^2/ν, where L is the distance from the wall of the tank. Thus, in one spin-down time diffusion penetrates a distance $L = (293\nu)^{1/2} = 1.7$ cm.

8.5. Suppose that at 43°N the geostrophic wind is westerly at $15\,\mathrm{m\,s^{-1}}$. Compute the net cross isobaric transport in the planetary boundary layer using both the mixed layer solution (8.22) and the Ekman layer solution (8.31). You may let $|\mathbf{V}| = u_g$ in (8.22), $h = De = 1$ km, $\kappa_s = 0.05\,\mathrm{m^{-1}\,s}$, and $\rho = 1\,\mathrm{kg\,m^{-3}}$.

Solution: For the mixed layer case the cross isobaric mass flux is $M = \rho\overline{v}h\,\mathrm{kg\,m^{-1}}$. But from (8.22) $\overline{v} = \kappa_s\left|\overline{\mathbf{V}}\right|u_g/\left(1 + \kappa_s^2\left|\overline{\mathbf{V}}\right|^2\right) = \kappa_s u_g^2/\left(1 + \kappa_s^2 u_g^2\right) = 7.2\,\mathrm{m\,s^{-1}}$, so mixed layer case gives $M = \rho\overline{v}h = 7.2\times10^3\,\mathrm{kg\,m^{-1}\,s^{-1}}$. For the Ekman layer, $\gamma = \pi/De$, so $M = \int_0^{De}\rho v dz = \int_0^{De}\rho u_g e^{-\gamma z}\sin\gamma z dz = \rho u_g(2\gamma)^{-1}(1 + e^{-\pi}) = 2.49\times 10^3\,\mathrm{kg\,m^{-1}\,s^{-1}}$.

8.6. Derive an expression for the wind-driven surface Ekman layer in the ocean. Assume that the wind stress τ_w is constant and directed along the x axis. Continuity of turbulent momentum flux at the air-sea interface ($z = 0$) requires the wind stress divided by air density must equal the oceanic turbulent momentum flux at $z = 0$. Thus, if the flux-gradient theory is used the boundary condition at the surface becomes $\rho_0 K \partial u/\partial z = \tau_w$, $\rho_0 K \partial v/\partial z = 0$, at $z = 0$, where K is the eddy viscosity in the ocean (assumed constant). As a lower boundary condition assume that $u, v \to 0$ as $z \to -\infty$. If $K = 10^{-3}\ \mathrm{m^2\ s^{-1}}$ what is the depth of the surface Ekman layer at 43°N latitude?

Solution: For $u_g = v_g = 0$, the general solution of (8.20) is $u + iv = A\exp\left[(if/K)^{1/2} z\right] + B\exp\left[-(if/K)^{1/2} z\right]$. But, $u + iv \to 0$ as $z \to -\infty$, which gives $B = 0$. From the surface boundary condition, $\left(\frac{\partial u}{\partial z}\right)_{z=0} = \frac{\tau_w}{\rho K} = \mathrm{Re}\left[A\,(if/K)^{1/2}\right]$ and $\left(\frac{\partial v}{\partial z}\right)_{z=0} = 0 = \mathrm{Im}\left[A(if/K)^{1/2}\right]$. But, $\sqrt{i} = [\exp(i\pi/2)]^{1/2} = \exp(i\pi/4) = (1+i)/\sqrt{2}$. Thus, the condition on $(\partial v/\partial z)_{z=0}$ implies that $A_{re} = -A_{im}$, while the condition on $(\partial u/\partial z)_{z=0}$ implies that $A_{re} = (\tau_w/2\rho K\gamma)$, where $\gamma = (f/2K)^{1/2}$.
$\therefore u = A_{re}e^{\gamma z}(\cos\gamma z + \sin\gamma z);\quad v = A_{re}e^{\gamma z}(-\cos\gamma z + \sin\gamma z)$.
Here $De = (\pi/\gamma) = \pi(2 \times 10^{-3}/10^{-4})^{1/2} = 14.05\,\mathrm{m}$.

8.7. Show that the vertically integrated mass transport in the wind-driven oceanic surface Ekman layer is directed 90° to the right of the surface wind stress in the Northern Hemisphere. Explain this result physically.

Solution: The only forces acting on the water column are the wind stress and the Coriolis force. Thus, the Coriolis force on the water column must oppose the wind stress vector, so the mass transport must be 90° to the right of the surface wind stress. This can be verified by considering the solution to Problem 8.6:

$$\int_{-\infty}^{0} \rho u dz = \int_{-\infty}^{0} \rho A_{re} e^{\gamma z}(\cos\gamma z + \sin\gamma z)dz = 0, \quad \text{and}$$

$$\int_{-\infty}^{0} \rho v dz = \int_{-\infty}^{0} \rho A_{re} e^{\gamma z}(-\cos\gamma z + \sin\gamma z)dz = -(\tau_w/f)\,\mathrm{kg\,m^{-1}\,s^{-1}}$$

8.8. A homogeneous barotropic ocean of depth H = 3 km has a zonally symmetric geostrophic jet whose profile is given by the expression $u_g = U\exp[-(y/L)^2]$, where $U = 1\,\mathrm{m\,s^{-1}}$ and $L = 200\,\mathrm{km}$ are constants. Compute the vertical velocity produced by convergence in the Ekman layer at the ocean bottom and show that the meridional profile of the secondary cross-stream motion $(\overline{v}, \overline{w})$ forced in the interior is the same as the meridional profile of u_g. What are the maximum values of $\overline{v}$ and $\overline{w}$ if $K = 10^{-3}\ \mathrm{m^2\ s^{-1}}$ and $f = 10^{-4}\ \mathrm{s^{-1}}$? (Assume that $\overline{w}$ and the eddy stress vanish at the surface.)

Solution: $\overline{w}_{De} = \zeta_g(K/2f)^{1/2} = -(\partial u_g/\partial y)\,(K/2f)^{1/2} = \left(2yU/L^2\right)(K/2f)^{1/2}\exp(-y^2/L^2)$. Maximum vertical velocity occurs at y, for which $\partial w_{De}/\partial y = 0$, or for $y = L/\sqrt{2}$. $(\overline{w}_{De})_{\max} = \left(\frac{\sqrt{2}}{2\times 10^5}\right)\sqrt{5}\exp(-1/2) = 9.6 \times 10^{-3}\,\mathrm{mm\,s^{-1}}$. From continuity equation for the fluid interior $\int_{De}^{H}\left(\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y}\right)dz = -\int_{De}^{H}\left(\frac{\partial \overline{w}}{\partial z}\right)dz = -(\overline{w}_H - \overline{w}_{De})$. But since $\overline{w}_H = 0$, $\overline{u}$ is independent of x, and $H \gg De$, this simplifies to $\int_0^H\left(\frac{\partial \overline{v}}{\partial y}\right)dz = \overline{w}_{De}$
$\therefore \overline{v} = \frac{1}{H}\int \overline{w}_{De}dy = -\left(\frac{U}{H}\right)\left(\frac{K}{2f}\right)^{1/2}\exp\left[-\left(\frac{y}{L}\right)^2\right]$. Thus, $\overline{v}_{\max} = -(3000)^{-1}(5)^{1/2} = -0.75\,\mathrm{mm\,s^{-1}}$ at $y = 0$.

8.9. Using the approximate zonally averaged momentum equation $\partial\overline{u}/\partial t \cong f\overline{v}$, compute the spin-down time for the zonal jet in Problem 8.8.

Solution: $\partial\overline{u}_g/\partial t \cong -(f/H)\,(K/2f)^{1/2}\,\overline{u}_g = -\tau^{-1}\overline{u}_g$. Integrating in time gives $\overline{u}_g(t) = \overline{u}_g(0)\exp(-t/\tau)$. $\tau = (H/f)\,(2f/K)^{1/2} = 1.34 \times 10^7$ s, or about 155 days.

8.10. Derive a formula for the vertical velocity at the top of the planetary boundary layer using the mixed layer expression (8.22). Assume that $|\mathbf{V}| = 5\ \mathrm{m\ s^{-1}}$ is independent of x and y and that $\overline{u}_g = \overline{u}_g(y)$. If $h = 1\,\mathrm{km}$ and $\kappa_s = 0.05$, what value must K_m have if this result is to agree with the vertical velocity derived from the Ekman layer solution at 43° latitude with $De = 1\,\mathrm{km}$?

Solution: From (8.22) $\overline{u} = \overline{u}_g \left(1 + \kappa_s^2 \left|\overline{\mathbf{V}}\right|^2\right)^{-1}$; $\overline{v} = \kappa_s \left|\overline{\mathbf{V}}\right| \overline{u}_g \left(1 + \kappa_s^2 \left|\overline{\mathbf{V}}\right|^2\right)^{-1}$, and from the continuity equation $w(h) = -\int_0^h \left(\frac{\partial \overline{v}}{\partial y}\right) dz = \frac{-\kappa_s |\overline{\mathbf{V}}| h}{\left(1+\kappa_s^2 |\overline{\mathbf{V}}|^2\right)} \frac{\partial \overline{u}_g}{\partial y} = \frac{\kappa_s |\overline{\mathbf{V}}| h}{\left(1+\kappa_s^2 |\overline{\mathbf{V}}|^2\right)} \overline{\zeta}_g$. For equality with the Ekman layer result, (8.38) must have $\frac{\kappa_s |\overline{\mathbf{V}}| h}{\left(1+\kappa_s^2 |\overline{\mathbf{V}}|^2\right)} = \left(\frac{K_m}{2f}\right)^{1/2} = 235$ m. Solving for K_m gives $K_m = 11\ \text{m}^2\ \text{s}^{-1}$.

8.11. Show that $K_m = kzu_*$ in the surface layer.

Solution: From (8.32) $K_m \partial \overline{u}/\partial z = u_*^2$. But, $\partial \overline{u}/\partial z = u_*/kz$. Thus $K_m = kzu_*$.