Chapter 7

Baroclinic Development

7.1. Show using Eq. (7.25) that the maximum growth rate for baroclinic instability when $\beta = 0$ occurs for $k^2 = 2\lambda^2\left(\sqrt{2}-1\right)$. How long does it take the most rapidly growing wave to amplify by a factor of e^1 if $\lambda^2 = 2\times 10^{-12}\ \mathrm{m^{-2}}$, and $U_T = 20\ \mathrm{m\ s^{-1}}$?

Solution: The maximum α occurs at the k for which $\partial\alpha/\partial k = 0$. Thus, $\frac{\partial\alpha^2}{\partial k} = 2\alpha\frac{\partial\alpha}{\partial k} = 2kU_T^2\left[\left(\frac{2\lambda^2-k^2}{2\lambda^2+k^2}\right) - \frac{k^2}{(2\lambda^2+k^2)} - k^2\frac{(2\lambda^2-k^2)}{(2\lambda^2+k^2)^2}\right] = 0$, which implies that $k^4 + 4k^2\lambda^2 - 4\lambda^4 = 0$. Thus, $k_{\max}^2 = 2\lambda^2\left(\sqrt{2}-1\right)$, and $\alpha_{\max} = k_{\max}U_T\left(\frac{2\lambda^2-k_{\max}^2}{2\lambda^2+k_{\max}^2}\right)^{1/2} = \sqrt{2}\lambda U_T\left(\sqrt{2}-1\right)$. For the values given above for U_T, and λ, $\alpha_{\max} = 1.66\times 10^{-5}\ \mathrm{s^{-1}}$, and the e^1 amplification time is $\alpha_{\max}^{-1} = 6.04\times 10^4$ s, or about 17 hours.

7.2. Solve for ψ_3' and ω_2' in terms of ψ_1' for a baroclinic Rossby wave whose phase speed satisfies (7.24). Explain the phase relationship between ψ_1', ψ_3', and ω_2' in terms of the quasi-geostrophic theory. (Note that $U_T = 0$ in this case.)

Solution: Here, $c - U_m = -\beta/\left(k^2+2\lambda^2\right)$ and $U_1 = U_3 = U_m$. Then, from (7.19) and (7.20), $A = 0$ so that $\psi_1' = -\psi_3'$. Then, from (7.12), $\left(\frac{\partial}{\partial t} + U_m\frac{\partial}{\partial x}\right)\left(\psi_1' - \psi_3'\right) = \frac{\sigma\delta p}{f_0}\omega_2'$.
Letting $\psi_1' = -\psi_3' = A\exp\left[ik(x-ct)\right]$ and $\omega_2' = C\exp\left[ik(x-ct)\right]$, substitution gives $C = \frac{ikf_0(U_m-c)}{\sigma\delta p}(2A)$, or $C = \frac{2ikf_0}{\sigma\delta p}\frac{\beta A}{(k^2+2\lambda^2)}$. Thus, if it is assumed that A is real, $\psi_1' = -\psi_3' = A\cos\left[k(x-ct)\right]$, and $\omega_2' = \frac{-2kf_0\beta A}{\sigma\delta p(k^2+2\lambda^2)}\sin\left[k(x-ct)\right]$.
Thus, ψ_1' and ψ_3' are 180° out of phase, and the maximum ω_2' occurs to the west of the maximum ψ_1' by 90° phase. This implies that the maximum *downward* motion occurs 90° west of the 250 hPa ridge and 750 hPa trough, while the maximum *upward* motion is 90° east of the 250 hPa ridge and 750 hPa trough. The convergence-divergence pattern thus generates vorticity changes that partly cancel the planetary vorticity advection so that the speed c of motion of the system is less than that given by planetary vorticity alone ($c = -\beta/k^2$).

7.3. For the case $U_1 = -U_3$ and $k^2 = \lambda^2$, solve for ψ_3' and ω_2' in terms of ψ_1' for marginally stable waves [i.e., $\delta = 0$ in (7.22)].

Solution: Here, $U_m = 0$ and $U_1 = -U_3 = U_T$, but since $\delta = 0$ and $\lambda^2 = k^2$, (7.22) gives $c = -\frac{2\beta}{3\lambda^2}$ and $U_T = \frac{\beta}{\sqrt{3}\lambda^2}$. Then, from (7.20), $\left[-\frac{2\beta}{3\lambda^2}\left(3\lambda^2\right) + \beta\right]B = -U_T\lambda^2 A$, or $B = A/\sqrt{3}$. Hence, $\psi_1' - \psi_3' = \left(\psi_1' + \psi_3'\right)/\sqrt{3}$, or $\psi_3' = \psi_1'(\sqrt{3}-1)/(\sqrt{3}+1)$, so that ψ_1' and ψ_3' are in phase. From (7.10), if we again let $\psi_1' = A\exp\left[ik(x-ct)\right]$ and $\omega_2' = C\exp\left[ik(x-ct)\right]$, we get $ik\left(U_T - c\right)\left(-k^2A\right) + ik\beta A = \left(f_0/\delta p\right)C$. Substituting for $U_T, k,$ and c, this yields: $C = \frac{-i\delta p}{f_0}\lambda\beta\left(\frac{\sqrt{3}-1}{3}\right)A$, and $\omega_2' = \frac{\delta p}{f_0}\lambda\beta\left(\frac{\sqrt{3}-1}{3}\right)A\sin\left[k(x-ct)\right]$. The maximum upward motion occurs 90° west of the ridge. The divergence-convergence field partly cancels vorticity advection at both the 250 hPa and 750 hPa levels, since the advecting winds are westerly at the upper level and easterly at the lower level.

7.4. For the case $\beta = 0$, $k^2 = \lambda^2$, and $U_m = U_T$, solve for ψ_3' and ω_2' in terms of ψ_1'. Explain the phase relationships between ω_2', ψ_1', and ψ_3' in terms of the energetics of quasi-geostrophic waves for the amplifying wave.

Solution: In this case $U_1 = 2U_m$, and $U_3 = 0$. From (8.25): $c = U_m\left(1 + i/\sqrt{3}\right)$. Let $\psi_m = A\exp\left[ik(x-ct)\right]\psi_T = B\exp\left[ik(x-ct)\right]$, $\omega_2' = C\exp\left[ik(x-ct)\right]$, $\psi_1' = E\exp\left[ik(x-ct)\right]$, $\psi_3' = F\exp\left[ik(x-ct)\right]$, where *A, B, C, E,* and *F*

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are all complex coefficients. From (7.16), then $k^2(c-U_m)A - k^2U_TB = 0$, or $B = \left(i/\sqrt{3}\right)A$. Thus, $(E-F) = i(E+F)/\sqrt{3}$ and $F = E\frac{\left(1-i/\sqrt{3}\right)}{\left(1+i/\sqrt{3}\right)} = E\left(1-i\sqrt{3}\right)/2$. But, $\psi_1' = \psi_m + \psi_T$ and $\psi_3' = \psi_m - \psi_T$, which implies that $A+B=E$ and $A-B=F$. Thus, $\psi_1' = \text{Re}\left\{A\left(1+i/\sqrt{3}\right)\exp\left[ik(x-ct)\right]\right\}$, $\psi_3' = \text{Re}\left\{A\left(1-i/\sqrt{3}\right)\exp\left[ik(x-ct)\right]\right\}$. From (8.10). $C = \left(\frac{\delta p}{f_0}\right)ik^3U_m\left(-\frac{4}{3}\right)A$.

$\omega_2' = -\text{Re}\left\{\frac{\delta p}{f_0}k^3U_m\left(4i/3\right)A\exp\left[ik(x-ct)\right]\right\}$. Hence, ψ_3' leads ψ_1' by 60° and ω_2' leads ψ_3' by 60°. Hence, trough and ridge lines tilt to the west with height, and subsidence is centered 60° to east of ridge at 750 hPa.

7.5. Suppose that a baroclinic fluid is confined between two rigid horizontal lids in a rotating tank in which $\beta = 0$ but friction is present in the form of linear drag proportional to the velocity (i.e., $\mathbf{Fr} = -\mu\mathbf{V}$). Show that the two-level model perturbation vorticity equations in Cartesian coordinates can be written as

$$\left(\frac{\partial}{\partial t} + U_1\frac{\partial}{\partial x} + \mu\right)\frac{\partial^2\psi_1'}{\partial x^2} - \frac{f}{\delta p}\omega_2' = 0$$

$$\left(\frac{\partial}{\partial t} + U_3\frac{\partial}{\partial x} + \mu\right)\frac{\partial^2\psi_3'}{\partial x^2} + \frac{f}{\delta p}\omega_2' = 0,$$

where perturbations are assumed in the form given in (7.9). Assuming solutions of the form (7.18), show that the phase speed satisfies a relationship similar to (7.22) with β replaced everywhere by $i\mu k$, and that as a result the condition for baroclinic instability becomes $U_T > \mu\left(2\lambda^2 - k^2\right)^{-\frac{1}{2}}$.

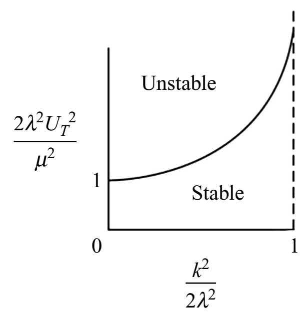


FIGURE 7.5

Solution: A linear drag of form $-\mu\mathbf{V}$ in the horizontal momentum equation will have the form $-\mu\mathbf{k}\cdot\nabla\times\mathbf{V} = -\mu\nabla^2\psi$ in the vorticity equation. Thus, for the two-level model with $\beta = 0$, the linearization proceeds as in the text with the addition of this term, and the results are as given above. Assuming that solutions exist of the form (7.18) gives

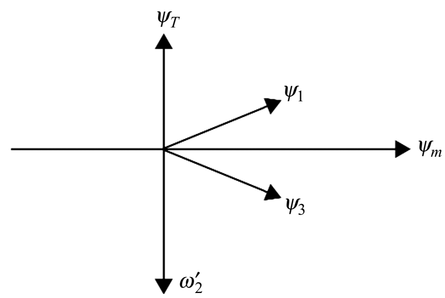
$$[ik(c-U_m) - \mu]k^2A - ik^3U_TB = 0$$

$$\left[ik(c-U_m)\left(k^2+2\lambda^2\right) - \mu k^2\right]B - ikU_T\left(k^2-2\lambda^2\right)A = 0.$$

These can be combined to yield a quadratic equation in c that is equivalent to (7.21) if we replace β by $i\mu k$. Thus, $c = U_m - i\mu\frac{(k^2+\lambda^2)}{[k(k^2+2\lambda^2)]} \pm \delta^{1/2}$, where $\delta = -\frac{\mu^2\lambda^4}{k^2(k^2+2\lambda^2)^2} - U_T^2\frac{(2\lambda^2-k^2)}{(k^2+2\lambda^2)}$. For instability c must have a positive imaginary part. Thus, δ must be negative and $\left|\delta^{1/2}\right| > \mu\frac{(k^2+\lambda^2)}{[k(k^2+2\lambda^2)]}$, or $|\delta| > \mu^2\frac{(k^2+\lambda^2)^2}{[k(k^2+2\lambda^2)]^2}$, which implies from the definition of δ that $U_T^2 = \mu^2/\left(2\lambda^2-k^2\right)$ for marginal stability. Thus, the marginal stability curve satisfies $k^2/2\lambda^2 = 1 - \mu^2/\left(2U_T^2\lambda^2\right)$. Note that for $k\to 0$, $U_T^2 \to \mu^2/\left(2\lambda^2\right)$ on the marginal curve and for $k^2 \to 2\lambda^2$, $U_T^2 \to \infty$ on the marginal curve.

7.6. For the case $\beta = 0$ determine the phase difference between the 250-mb and 750-mb geopotential fields for the most unstable baroclinic wave (see Problem 7.1). Show that the 500-mb geopotential and thickness fields are 90° out of phase.

Solution: For $k^2 = 2\lambda^2\left(\sqrt{2}-1\right)$ equation (8.25) gives $c - U_m = iU_T\left(\sqrt{2}-1\right)^{1/2}$. From (7.19) we then have $\left[(c-U_m)k^2\right]A = \left[iU_T\left(\sqrt{2}-1\right)^{1/2}k^2\right]A = k^2U_TB$. Thus, $B = i\left(\sqrt{2}-1\right)^{1/2}A$, which from (7.18) implies that if A is real, $\psi_m = A\cos\left[k\left(x-ct\right)\right]$ and $\psi_T = -A\left(\sqrt{2}-1\right)^{1/2}\sin\left[k\left(x-ct\right)\right]$ so that ψ_T lags ψ_m by 1/4 cycle (coldest air is 90° west of the trough at 500 hPa). Now, let $\psi_1' = \mathrm{Re}\left\{\hat{\psi}_1\exp\left[ik\left(x-ct\right)\right]\right\}$; $\psi_3' = \mathrm{Re}\left\{\hat{\psi}_3\exp\left[ik\left(x-ct\right)\right]\right\}$. Then from the above, $\hat{\psi}_1 - \hat{\psi}_3 = i\left(\sqrt{2}-1\right)^{1/2}\left(\hat{\psi}_1+\hat{\psi}_3\right)$, or $\hat{\psi}_3 = \hat{\psi}_1\left[\left(\sqrt{2}-1\right) - i\sqrt{2}\left(\sqrt{2}-1\right)^{1/2}\right]$, from which, by noting that the tangent of the phase is the ratio of imaginary to real parts, we get $\phi = \tan^{-1}\left[\frac{\sqrt{2}\left(\sqrt{2}-1\right)^{1/2}}{\left(\sqrt{2}-1\right)}\right] = 65.5°$. So the ψ_3' field leads ψ_1' by 65.5°.



7.7. For the conditions of Problem 7.6, given that the amplitude of ψ_m is $A = 10^7$ m^2 s^{-1}, solve (7.19)–(7.20) to obtain B. Let $\lambda^2 = 2\times10^{-12}$ m^{-2} and $U_T = 15$ m s^{-1}.

Solution: From problem 7.6: $|B| = \left|\left(\sqrt{2}-1\right)^{1/2}A\right| = 6.44\times10^6$ m^2 s^{-1}. Thus, the amplitude of B does not depend on U_T or λ^2 in this case, and further $\left|\psi_1'\right| = \left|\psi_3'\right|$.

7.8. For the situation of Problem 7.7, compute ω_2' using the expression (8.29).

Solution: Now, $\zeta_2 = \nabla^2\psi_m$. Thus, if we let $\psi_m = A\exp\left[ik\left(x-ct\right)\right]$, and $\omega_2' = C\exp\left[ik\left(x-ct\right)\right]$, then substitution into (8.29) gives $-(k^2+2\lambda^2)C = \left(4f_0/\sigma\delta p\right)U_Tk^2\left(ik\right)A$, which after lettting $k^2 = 2\lambda^2\left(\sqrt{2}-1\right)$ yields $C = -(i4f_0/\sigma\delta p)\lambda(\sqrt{2}-1)^{3/2}U_TA$. Thus, if A is real so that $\psi_m = A\cos\left[k\left(x-ct\right)\right]$, then $\omega_2' = \left(4f_0/\sigma\delta p\right)\lambda(\sqrt{2}-1)^{3/2}U_TA\sin[k(x-ct)]$, so maximum upward motion is 90° in phase to east of 500-hPa trough.

7.9. Compute the total potential energy per unit cross-sectional area for an atmosphere with an adiabatic lapse rate given that the temperature and pressure at the ground are $p = 10^5$ Pa and $T = 300$ K, respectively.

Solution: $E_P + E_I = \frac{c_p}{c_v}E_I = \frac{c_p}{c_v}\int_0^\infty(\rho c_vT)dz = \frac{c_p}{R}\int_0^\infty p\,dz$. Now, for an adiabatic lapse rate the potential temperature is constant and equal to the temperature T_0 at $p_s = 1000$ hPa. Thus, since by definition $\theta = T(p_s/p)^{R/c_p}$, the pressure dependence on z in this case is $\left(\frac{p}{p_s}\right) = \left(\frac{T}{T_0}\right)^{c_p/R} = \left(\frac{T_0+\Gamma z}{T_0}\right)^{c_p/R}$, where $\Gamma = -g/c_p$. Thus, total potential energy is given by $\frac{c_p}{c_v}E_I = \frac{c_p}{R}\int_0^{z_T}p_s\left(\frac{T_0+\Gamma z}{T_0}\right)^{c_p/R}dz$, where $z_T = -T_0/\Gamma = c_pT_0/g$ is the height of the adiabatic atmosphere. From the above integral $\frac{c_p}{c_v}E_I = \frac{c_p}{R}p_sT_0\left(\frac{T_0+\Gamma z}{T_0}\right)^{\frac{c_p}{R}+1}\left(\frac{\Gamma^{-1}}{c_p/R+1}\right)\Bigg|_0^{z_T} = \left(\frac{p_sT_0}{g}\right)\left(\frac{c_p}{1+R/c_p}\right) = 2.38\times10^9$ J m^{-2}.

7.10. Consider two air masses at the uniform potential temperatures $\theta_1 = 320$ K and $\theta_2 = 340$ K, which are separated by a vertical partition as shown in Figure 7.7. Each air mass occupies a horizontal area of $A = 10^4$ m^2 and extends from the surface ($p_0 = 10^5$ Pa) to the top of the atmosphere. What is the available potential energy for this system? What fraction of the total potential energy is available in this case?

Solution: Using the results of problem 7.9, we have for the total energy in the initial state: $\left(\frac{p_s}{g}\right)\frac{c_pA}{(1+R/c_p)}(\theta_1+\theta_2) = \frac{(10^5)(1004)(10^4)}{(9.8)(9/7)}(320+340) = 5.259\times10^{13}$ J. To compute the energy in the final state of minimum total potential energy (with the warm air on top of the cold air) note that with the aid of the hydrostatic equation the total energy can be expressed as an integral of temperature with respect to pressure: $\frac{c_p}{c_v}E_I = c_p\int_0^\infty(\rho T)dz = -\frac{c_p}{g}\int_{p_s}^0 Tdp$. But $T = \theta\left(\frac{p}{p_s}\right)^{R/c_p}$. Thus, for unit area $\frac{c_p}{c_v}E_I = -2A\left[\frac{c_p}{g}\int\limits_{p_s}^{p_s/2}\theta_1\left(\frac{p}{p_s}\right)^{R/c_p}dp + \frac{c_p}{g}\int\limits_{p_s/2_s}^{0}\theta_2\left(\frac{p}{p_s}\right)^{R/c_p}dp\right]$, where the $2A$ factor comes from multiplying energy per unit area times the total area. Evaluating these integrals yields $\frac{2Ac_pp_s}{g(1+R/c_p)}\left[\theta_1\left(1-0.5^{9/7}\right)+\theta_2\left(0.5^{9/7}\right)\right] = 5.23\times10^{13}$ J. But available potential energy is the difference between the initial and final total potential energies $= 5.259\times10^{13} - 5.230\times10^{13} = 2.9\times10^{11}$ J. The percent of total potential energy that is available is $\left(0.029/5.259\right)\times100 = 0.55\%$.

7.11. For the unstable baroclinic wave that satisfies the conditions given in Problems 7.6, 7.7, and 7.8, compute the energy conversion terms in (8.38) and (8.39) and hence obtain the instantaneous rates of change of the perturbation kinetic and available potential energies.

Solution: From (7.38) $dK'/dt = -(2f_0/\delta p)\overline{\omega_2'\psi_T}$. Let $\psi_m = A\cos[k(x-ct)]$. Then from Problem 7.6 $\psi_T = -(\sqrt{2}-1)^{1/2}A\sin[k(x-ct)]$ and from Problem 8.8, $\omega_2' = \left(4f_0/\sigma\delta p\right)\lambda(\sqrt{2}-1)^{3/2}U_TA\sin[k(x-ct)]$. But $\overline{\sin^2[k(x-ct)]} = 0.5$, so that $\overline{\omega_2'\psi_T} = -\left(\sqrt{2}-1\right)^2A^2\left(2f_0/\sigma\delta p\right)\lambda U_T$. Thus, $dK'/dt = 4\left(\sqrt{2}-1\right)^2A^2\lambda^3U_T$, where we have used the fact that $\lambda^2 = f_0^2/\left(\sigma\delta p^2\right)$. For $\lambda = 1.2\times10^{-6}\,\text{m}^{-1}$, $U_T = 15\,\text{m}\,\text{s}^{-1}$, and $A = 10^7\,\text{m}^2\,\text{s}^{-1}$, $dK/dt = 1.78\times10^{-3}\,\text{J}\,\text{kg}^{-1}\,\text{s}^{-1}$.

Similarly, from (7.39) the eddy potential energy generation term is $4\lambda^2U_T\overline{\psi_T\left(\partial\psi_m/\partial x\right)} = 2\lambda^2U_TA^2k\left(\sqrt{2}-1\right)^{1/2} = 2\lambda^3U_TA^2\sqrt{2}\left(\sqrt{2}-1\right) = 3.04\times10^{-3}\,\text{J}\,\text{kg}^{-1}\,\text{s}^{-1}$. Thus, $dP'/dt = 3.04\times10^{-3} - 1.78\times10^{-3} = 1.26\times10^{-3}\,\text{J}\,\text{kg}^{-1}\,\text{s}^{-1}$.

7.12. Starting with (7.62) and (7.64) derive the phase speed c for the Eady wave given in (7.70).

Solution: Substituting from (7.67) into (7.64) and (7.65) gives: $ik(\Lambda z-c)\left[-\left(k^2+l^2\right)\Psi+\varepsilon\frac{d^2\Psi}{dz^2}\right] = 0$, and $ik(\Lambda z-c)\frac{d\Psi}{dz} - ik\Psi\Lambda = 0$. From these (7.68) follows immediately. Substituting the general solution (7.69) into (7.68) at z =0 gives $B = -(c\alpha/\Lambda)A$. Using this relation and substituting (7.69) into (7.68) at $z = H$ then gives $\alpha(\Lambda H-c)\left(\cosh\alpha H - \frac{c\alpha}{\Lambda}\sinh\alpha H\right) - \Lambda(\sinh\alpha H - \frac{c\alpha}{\Lambda}\cosh\alpha H) = 0$, which can be rewritten as a quadratic in c: $c^2 - \Lambda Hc + \left(\frac{\Lambda^2H}{\alpha}\frac{\cosh\alpha H}{\sinh\alpha H} - \frac{\Lambda^2}{\alpha^2}\right) = 0$, from which the solution for phase speed given in (7.70) follows.

7.13. Unstable baroclinic waves play an important role in the global heat budget by transferring heat poleward. Show that for the Eady wave solution the poleward heat flux averaged over a wavelength

$$\overline{v'T'} = \frac{1}{L}\int\limits_0^L v'T'dx$$

is independent of height and is positive for a growing wave. How does the magnitude of the heat flux at a given instant change if the mean wind shear is doubled?

Solution: We can let $\psi' = \Psi(z)\cos ly\exp[ik(x-ct)]$. Thus, for a growing wave $\psi' = [\Psi_r\cos k(x-c_rt) - \Psi_i\sin k(x-c_rt)](\cos ly)\exp(kc_it)$. This implies that $v' = \partial\psi'/\partial x = [-\Psi_rk\sin k(x-c_rt) - \Psi_ik\cos k(x-c_rt)](\cos ly)$ $\exp(kc_it)$, and $T' \propto \partial\psi'/\partial z = \left[\frac{d\Psi_r}{dz}\cos k(x-c_rt) - \frac{d\Psi_i}{dz}\sin k(x-c_rt)\right](\cos ly)\exp(kc_it)$. Then $\overline{v'T'} \propto -\Psi_i\frac{d\Psi_r}{dz} + \Psi_r\frac{d\Psi_i}{dz}$

(since $\overline{\sin^2 kx} = \overline{\cos^2 kx} = 1/2$, and $\overline{\sin kx \cos kx} = 0$). But from (7.70) $\Psi(z) = A \sinh \alpha z + B \cosh \alpha z$, and $B = -(c\alpha/\Lambda)A$. Thus, if we let A be real: $\Psi_r = A\left[\sinh \alpha z - (c_r\alpha/\Lambda)\cosh \alpha z\right]$ and $\Psi_i = -(c_i\alpha/\Lambda)A\cosh \alpha z$. Thus, $-\Psi_i \frac{d\Psi_r}{dz} + \Psi_r \frac{d\Psi_i}{dz} = \frac{c_i\alpha^2 A^2}{\Lambda}\left(\cosh^2 \alpha z - \sinh^2 \alpha z\right) = \frac{c_i\alpha^2 A^2}{\Lambda}$, which is positive for $c_i > 0$, and is independent of height. Furthermore, since from (7.71) $c_i \propto \Lambda$, we confirm that $\overline{v'T'}$ does not depend on the magnitude of the shear.

7.14. Assuming that the coefficient A in (7.69) is real obtain an expression for the geostrophic streamfunction $\psi'(x, y, z^*, t)$ for the most unstable mode in the Eady stability problem for the case $k = m$. Use this result to derive an expression for the corresponding vertical velocity w^* in terms of A.

Solution: The wavelength of maximum growth for $k = l$ is given in the text below equation (7.72). From this result we find $\alpha_m \cong 1.6$. From the lower boundary condition we immediately find $B = -c\alpha A/\Lambda$, where $c = c_r + ic_i$. Then from (7.67) and (7.70) the streamfunction is

$$\begin{aligned}
&\psi\left(x, y, z^*, t\right) = \mathrm{Re}\left\{A\left(\sinh \alpha z^* - c\alpha/\Lambda \cosh \alpha z^*\right)\cos ly \exp\left[ik\left(x - ct\right)\right]\right\}, \quad \text{or} \\
&\psi\left(x, y, z^*, t\right) = A \cos ly \exp\left(kc_i t\right)\left[G_r \cos k\left(x - c_r t\right) - G_i \sin k\left(x - c_r t\right)\right], \quad \text{where} \\
&G_r = \sinh \alpha_m z - \left(c_r\alpha_m/\Lambda\right)\cosh \alpha_m z; \quad G_i = -\left(c_i\alpha_m/\Lambda\right)\cosh \alpha_m z
\end{aligned}$$

Substitution into (7.65) then yields for the vertical velocity:

$$w^* = \frac{f_0}{N^2} A \mathrm{Re}\left\{ik\left[-\alpha\Lambda z^* \cosh \alpha z^* - \left(c^2 - c\Lambda z^* - \frac{\Lambda^2}{\alpha^2}\right)\frac{\alpha^2}{\Lambda}\sinh \alpha z^*\right]\exp\left[ik\left(x - ct\right)\right]\right\}.$$

7.15. For the neutral baroclinic wave disturbance in the two-layer model given by (7.75a,b) derive the corresponding ω_2' field. Describe how the convergence and divergence fields associated with this secondary circulation influence the evolution of the disturbance.

Solution: From (7.29), (7.75a) and the definition of λ, we can write

$$\begin{aligned}
\left(\frac{\partial^2}{\partial x^2} - 2\lambda^2\right)\omega_2' &= -\frac{\delta p}{f_0}\left(4\lambda^2 U_T \frac{\partial^3 \psi_m}{\partial x^3}\right) \\
&= -\frac{\delta p}{f_0} 4\lambda^2 U_T \left(\frac{2\mu A_1 k^3}{1 + \mu}\right)\left[\sin kx \cos\left(k\mu U_T t\right) - \frac{1}{\mu}\cos kx \sin(k\mu U_T t)\right].
\end{aligned}$$

Thus, $\omega_2' = \frac{\delta p}{f_0}\frac{8\lambda^2 U_T \mu A_1 k^3}{(k^2+2\lambda^2)(1+\mu)}\left[\sin kx \cos(k\mu U_T t) - \frac{1}{\mu}\cos kx \sin(k\mu U_T t)\right]$.
During growth phase upward motion is centered east of the upper level trough, so convergence at the lower level concentrates negative vorticity east of initial trough and leads to development at the lower level. At time of maximum amplification upward motion is 90° east of the trough and the divergence-convergence pattern cancels vorticity advection at both upper and lower layers, so the system remains stationary.

7.16. For the situation of Problem 7.15 derive expressions for the conversion of zonal available potential energy to eddy available potential energy and the conversion of eddy available potential energy to eddy kinetic energy.

Solution: From (7.39) the conversion of mean to eddy available potential energy is given by $4\lambda^2 U_T \overline{\psi_T \partial\psi_m/\partial x}$. Substituting from (7.75) then yields $\frac{d(P'+K')}{dt} = 8\lambda^2 U_T \mu k A_1^2\left(\frac{1-\mu}{1+\mu}\right)\sin(k\mu U_T t)\cos(k\mu U_T t)$. The conversion from eddy available potential energy to eddy kinetic energy is given from (7.39) as $2\frac{f_0}{\delta p}\overline{\omega_2'\psi_T}$. Substituting from (7.75) and the solution of Problem 7.15, we then get $\frac{dK'}{dt} = 8\lambda^2 U_T \mu k A_1^2\left(\frac{2k^2}{k^2+2\lambda^2}\right)\left(\frac{1-\mu}{1+\mu}\right)\sin(k\mu U_T t)\cos(k\mu U_T t)$. The ratio of the growth of total eddy energy to growth of eddy kinetic energy is thus $\left(k^2 + 2\lambda^2\right)/\left(2k^2\right) < 1$ for $k^2 > 2\lambda^2$. Thus, for neutral modes kinetic energy increases more rapidly than total eddy energy, indicating that eddy available potential energy decreases in time. This is consistent with the evolution toward a barotropic structure.