Chapter 6

Quasi-Geostrophic Analysis

6.1. Show that for a basic-state having linear shear, $\overline{U} = \lambda z$, λ constant, that the basic-state QG PV depends only on z, and that the linearized disturbance QG PV obeys

$$\left(\frac{\partial}{\partial t} + \overline{U}\frac{\partial}{\partial x}\right) q' = 0. \tag{1}$$

Solution: The QG PV is given by

$$q = \zeta_g + f + f\frac{\partial}{\partial z}\left(\frac{d\overline{\theta}}{dz}^{-1}\theta\right). \tag{2}$$

The basic state vorticity, $-\frac{\partial \overline{U}}{\partial y}$, is zero in this situation, so the basic-state QG PV takes the form

$$\overline{q} = f + f\frac{\partial}{\partial z}\left(\frac{d\overline{\theta}}{dz}^{-1}\overline{\theta}\right).$$

Thermal wind balance for the basic state reveals that

$$\frac{d\overline{U}}{dz} = -\frac{g}{f\overline{\theta}}\frac{\partial\overline{\theta}}{\partial y} = \lambda.$$

Integrating this result gives

$$\overline{\theta} = -\frac{\lambda f \overline{\theta}_r y}{g} + C,$$

where C is at most a function of z. Since $\zeta_g = 0$ and $\overline{\theta}$ is a function of z only, by (2), the QG PV is at most a function of z only as well.
From (6.16),

$$\frac{\partial q}{\partial t} + \mathbf{V}_g \cdot \nabla_h q = 0.$$

Since the basic state is a function of z only, we have

$$\frac{\partial q'}{\partial t} + \mathbf{V}_g \cdot \nabla_h q' = 0,$$

where primes denote perturbations from the basic state. Linearizing and noting that the basic state has only a zonal component, $\overline{U}$, gives

$$\frac{\partial q'}{\partial t} + \overline{U}\frac{\partial q'}{\partial x} = 0.$$

6.2. Show that the Boussinesq continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

can be written as

$$\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y} + \frac{\partial w}{\partial z} = 0.$$

Solution: The total wind may be expressed as a sum of the geostrophic and ageostrophic wind components: $u = u_g + u_a$ and $v = v_g + v_a$. Plugging into the continuity equation gives

$$\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y} + \frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{3}$$

Using the geostrophic wind definitions

$$u_g = -\frac{1}{\rho_0 f}\frac{\partial p}{\partial y} \quad v_g = \frac{1}{\rho_0 f}\frac{\partial p}{\partial x}$$

gives

$$\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y} = \frac{1}{\rho_0 f}\left(\frac{\partial^2 p}{\partial x \partial y} - \frac{\partial^2 p}{\partial x \partial y}\right) = 0$$

so that (3) becomes

$$\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y} + \frac{\partial w}{\partial z} = 0.$$

6.3. Prove identity (6.49):

$$-\frac{g}{\overline{\theta} N^2}\frac{\partial^2 v_j}{\partial x_i^2}\frac{\partial \theta}{\partial x_j} = \frac{f}{N^2}\frac{\partial v_j}{\partial x_3}\frac{\partial \zeta}{\partial x_j}. \tag{4}$$

Solution: Recall that the subscript "g" for geostrophic flow has been dropped here. Expanding the left-hand side gives

$$-\frac{g}{\overline{\theta} N^2}\frac{\partial^2 v_j}{\partial x_i^2}\frac{\partial \theta}{\partial x_j} = -\frac{g}{\overline{\theta} N^2}\left[\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)\frac{\partial \theta}{\partial x} + \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)\frac{\partial \theta}{\partial y}\right]. \tag{5}$$

Taking $\frac{\partial}{\partial x}$ of the quasi-geostrophic mass conservation equation (6.19) gives

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial xy} = 0. \tag{6}$$

This may be rewritten as

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial xy} + \frac{\partial^2 u}{\partial y^2} = -\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = -\frac{\partial \zeta}{\partial y}. \tag{7}$$

Similarly, $\frac{\partial}{\partial y}$ of the quasi-geostrophic mass conservation equation (6.19) gives

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{\partial \zeta}{\partial x}. \tag{8}$$

Using (7) and (8) in (5) gives

$$-\frac{g}{\overline{\theta} N^2}\frac{\partial^2 v_j}{\partial x_i^2}\frac{\partial \theta}{\partial x_j} = -\frac{g}{\overline{\theta} N^2}\left[-\frac{\partial \zeta}{\partial y}\frac{\partial \theta}{\partial x} + \frac{\partial \zeta}{\partial x}\frac{\partial \theta}{\partial y}\right]. \tag{9}$$

Using thermal wind,

$$\frac{\partial \theta}{\partial x} = -\frac{f}{\overline{\theta} g}\frac{\partial v}{\partial z} \qquad \frac{\partial \theta}{\partial y} = -\frac{f}{\overline{\theta} g}\frac{\partial u}{\partial z}$$

gives

$$-\frac{g}{\overline{\theta}N^2}\frac{\partial^2 v_j}{\partial x_i^2}\frac{\partial\theta}{\partial x_j} = \frac{f}{N^2}\left[\frac{\partial u}{\partial z}\frac{\partial\zeta}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial\zeta}{\partial y}\right] = \frac{f}{N^2}\frac{\partial v_i}{\partial x_3}\frac{\partial\zeta}{\partial x_i}. \tag{10}$$

6.4. In this problem you will derive an equation for the QG streamfunction of the ageostrophic secondary circulation. You should assume constant f and $\frac{d\overline{\theta}}{dz}$, and that the geostrophic flow is two dimensional (y–z); specifically: $u_g = u_g(z, t)$ only and $v_g = v_g(y, t)$ only.

(a) Starting with geostrophic and hydrostatic balance

$$\vec{V}_g = \frac{1}{\rho_0 f}\mathbf{k}\times\nabla_h p \quad \frac{\partial p}{\partial z} = \frac{\rho_0 g}{\overline{\theta}}\theta,$$

show that the maintenance of thermal wind balance requires

$$\frac{D}{Dt_g}\frac{\partial u_g}{\partial z} = -\frac{g}{\overline{\theta}f}\frac{D}{Dt_g}\frac{\partial\theta}{\partial y}.$$

(b) Determine the left-hand side of the result in (a) from the u_g momentum equation.

$$\frac{Du_g}{Dt_g} = f v_a$$

Interpret the result physically.

(c) Determine the right-hand side of the result in (a) from the thermodynamic energy equation. Interpret the result physically.

(d) Now use your results from (b) and (c) in (a) using a streamfunction for the ageostrophic wind: $v_a = \frac{\partial\psi}{\partial z}$ and $w = -\frac{\partial\psi}{\partial y}$. Express your result with all streamfunction terms on the left-hand side of the equation.

(e) Assume that the right-hand side of your result in (d) reaches a local minimum in the midtroposphere. Sketch the ageostrophic circulation streamlines in a y–z cross section along with arrows showing v_a and w.

Solution:

(a) By the thermal-wind equation

$$\frac{D}{Dt_g}\frac{\partial u_g}{\partial z} = \frac{D}{Dt_g}\left[-\frac{g}{f\overline{\theta}}\frac{\partial\theta}{\partial y}\right] = -\frac{g}{\overline{\theta}f}\frac{D}{Dt_g}\frac{\partial\theta}{\partial y}. \tag{11}$$

(b) Taking the vertical derivative of the zonal momentum equation gives

$$\frac{D}{Dt_g}\left(\frac{\partial u_g}{\partial z}\right) + \frac{\partial\mathbf{V}_g}{\partial z}\cdot\nabla_h u_g = f\frac{\partial v_a}{\partial z}. \tag{12}$$

Since u is a function of z and t, $\frac{\partial\mathbf{V}_g}{\partial z}\cdot\nabla_h u_g = 0$, and

$$\frac{D}{Dt_g}\left(\frac{\partial u_g}{\partial z}\right) = f\frac{\partial v_a}{\partial z}. \tag{13}$$

Following the geostrophic wind, changes in the shear of the vertical shear of the zonal wind are due to the vertical shear in the meridional ageostrophic wind.

(c) Taking the y derivative of the QG thermodynamic energy equation (6.8),

$$\frac{\partial}{\partial y}\frac{D\theta}{Dt_g} = -\frac{\partial w}{\partial y}\frac{d\overline{\theta}}{dz}. \tag{14}$$

Taking derivatives on the left side gives

$$\frac{D}{Dt_g}\left(\frac{\partial\theta}{\partial y}\right) + \frac{\partial\mathbf{V}_g}{\partial y}\cdot\nabla_h\theta = -\frac{\partial w}{\partial y}\frac{d\overline{\theta}}{dz}. \tag{15}$$

Noting that $\frac{\partial \mathbf{V}_g}{\partial y} \cdot \nabla_h \theta = \frac{\partial \theta}{\partial y}\frac{\partial v_g}{\partial y}$ gives

$$\frac{D}{Dt_g}\left(\frac{\partial \theta}{\partial y}\right) = -\frac{\partial w}{\partial y}\frac{d\overline{\theta}}{dz} - \frac{\partial \theta}{\partial y}\frac{\partial v_g}{\partial y}. \tag{16}$$

Multiplying by $-\frac{g}{\overline{\theta} f}$ gives

$$-\frac{g}{\overline{\theta} f}\frac{D}{Dt_g}\left(\frac{\partial \theta}{\partial y}\right) = \frac{N^2}{f}\frac{\partial w}{\partial y} + \frac{g}{\overline{\theta} f}\frac{\partial \theta}{\partial y}\frac{\partial v_g}{\partial y}, \tag{17}$$

where N is the buoyancy frequency. This result says that following the geostrophic wind, the meridional potential temperature gradient changes due to "tilting" of the ambient stratification into the meridional plane and "squeezing" of existing meridional gradients in potential temperature by the meridional geostrophic wind.

(d) Using the results from (b), (c), and (a) gives

$$f\frac{\partial v_a}{\partial z} = \frac{N^2}{f}\frac{\partial w}{\partial y} + \frac{g}{\overline{\theta} f}\frac{\partial \theta}{\partial y}\frac{\partial v_g}{\partial y}. \tag{18}$$

From the definition of the streamfunction

$$\frac{\partial v_a}{\partial z} = \frac{\partial^2 \psi}{\partial z^2} \qquad \frac{\partial w}{\partial y} = -\frac{\partial^2 \psi}{\partial y^2}, \tag{19}$$

so that (18) becomes

$$N^2\frac{\partial^2 \psi}{\partial y^2} + f^2\frac{\partial^2 \psi}{\partial z^2} = \frac{g}{\overline{\theta} f}\frac{\partial \theta}{\partial y}\frac{\partial v_g}{\partial y}. \tag{20}$$

(e) An example of where the right-hand side of (20) reaches a minimum is shown in Figure 6.1. The meridional geostrophic wind, shown in double arrows, acts to weaken the northward gradient in potential temperature, which is less than zero. The Laplacian-like operator for ψ reverses the sign of the solution, so that ψ reaches a local maximum near the local minimum in the "forcing." This produces a rising motion in the cold air and a sinking motion in the warm air, which reduces the weakening effect of the geostrophic flow on the potential temperature gradient. Moreover, the effect of v_a is to accelerate (decelerate) u_g at low (upper) levels, which weakens the westerly shear in accordance with the geostrophic flow tendency to weaken the meridional temperature gradient.

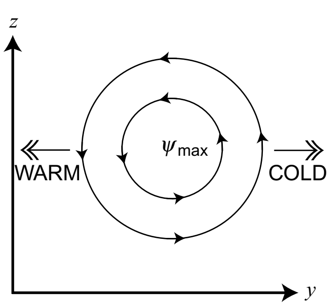

FIGURE 6.1 Streamfunction ageostrophic circulation. The geostrophic wind is indicated by double arrows and relative potential temperature values by "WARM" and "COLD." The streamfunction contours are given in solid lines, with the sense of the circulation indicated by arrows.

6.5. Consider two spherical, cyclonic, QG PV anomalies. One anomaly is located at the origin and has unity radius. The second anomaly is located at $(x, y, z) = (2, 0, 0)$ and has radius and amplitude one-half the corresponding values of the other anomaly. Draw an (x, y) sketch estimating the trajectory traced out by these two anomalies. Draw a second sketch showing the w field at a level above the anomalies.

Solution: The self-induced flow from each vortex has no effect on the vertices; all motion is due to flow from the other vortex. Since that flow is normal to the line connecting the vortex centers, the distance between them remains constant. As a result, both vortices trace out cyclonic (counterclockwise in the Northern Hemisphere and clockwise in the Southern

Hemisphere) trajectories. These trajectories take the form of circles about a center determined by the QG PV "center of mass"; qualitatively, this is located between the vortices, closer to the larger (Figure 6.2, left side).
Vertical motion is easily estimated by the Sutcliffe-form of the w equation,

$$\tilde{L}w = -\frac{2}{N^2}\left[-\mathbf{V}_{\mathrm{T}} \cdot (f\zeta_g)\right] + D, \tag{21}$$

where upward motion is associated with cyclonic vorticity advection by the thermal wind. Anomalously warm air is found above the PV anomalies, with a maximum near the anomaly (see Figure 6.10d). As a result, the thermal wind associated with each vortex is anticyclonic and centered on the vortex; in the Northern Hemisphere, the thermal wind flows clockwise around the PV anomalies. Appealing again to Figure 6.10d, above the PV anomaly, the potential temperature anomaly weakens with height, so the static stability contribution to the QG PV is negative; as a result, the relative vorticity must be cyclonic. Therefore, advection of vorticity by the thermal wind gives a pattern of rising and sinking air as shown in Figure 6.2 (right panel).

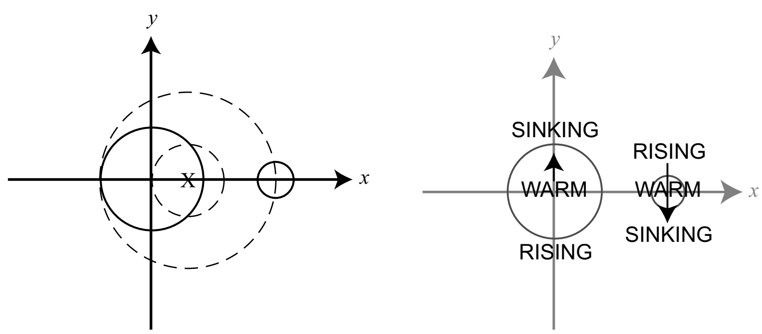

FIGURE 6.2

6.6. Derive the dispersion relationship for 3D QG Rossby waves on the β-plane. Assume plane waves in the absence of mean flow and boundaries. The governing (disturbance PV) equation is modified by an additional term

$$\frac{Dq}{Dt_g} + v\beta = 0.$$

(ii) Derive the group velocity vector. Explain the result.

Solution: The linearized governing equation is given by

$$\frac{\partial q'}{\partial t} + v'\beta = 0. \tag{22}$$

Assuming the wave solution form for the perturbation pressure,

$$p' = Re\left\{A\, e^{i(kx+ly+mz-\nu t)}\right\} \tag{23}$$

and using (6.24) gives

$$\frac{\partial q'}{\partial t} = \frac{i\nu}{\rho_0}\left(k^2 + l^2 + \frac{f^2}{N^2}m^2\right)p'. \tag{24}$$

From the geostrophic wind relationship

$$v' = \frac{ik}{\rho_0 f}p'. \tag{25}$$

Substituting (23) and (24) in (22) gives

$$\frac{i\nu}{\rho_0 f}\left(k^2 + l^2 + \frac{f^2}{N^2}m^2\right)p' + \frac{ik\beta}{\rho_0 f}p' = 0. \tag{26}$$

Solving for ν gives

$$\nu = \frac{-\beta k}{k^2 + l^2 + \frac{f^2}{N^2} m^2}. \tag{27}$$

(ii) Group velocity is defined by $\mathbf{C}_g = \nabla_K \nu = \left(\frac{\partial \nu}{\partial k}, \frac{\partial \nu}{\partial l}, \frac{\partial \nu}{\partial m}\right)$. Let

$$K = k^2 + l^2 + \frac{f^2}{N^2} m^2. \tag{28}$$

The scalar components of the group velocity are then given by

$$\frac{\partial \nu}{\partial k} = -\frac{\beta k}{kK} + 2\frac{k^2 \beta}{K^2} = \frac{\nu}{k} - 2\nu \frac{k}{K} \tag{29}$$

$$\frac{\partial \nu}{\partial l} = \frac{\beta k}{K^2} 2l = -\nu \frac{l}{K} \tag{30}$$

$$\frac{\partial \nu}{\partial m} = \frac{\beta k}{K^2} 2m = -\nu \frac{m}{K}. \tag{31}$$

Therefore,

$$\mathbf{C}_g = -2\frac{\nu}{K}\left(k - \frac{K}{2k}, l, m\right). \tag{32}$$

The group velocity is directed opposite to the phase speed, except when $k - \frac{K}{2k} < 0$, or

$$k^2 < l^2 + \frac{f^2}{N^2} m^2,$$

which applies to waves with sufficiently long wavelengths in the zonal direction compared to the meridional and vertical directions.

6.7. Derive the QG kinetic energy equation

$$\frac{\partial K}{\partial t_g} + \nabla \cdot \vec{S} = \frac{g}{\theta_0} w\theta$$

starting with the QG momentum equation

$$\frac{D\vec{V}_g}{Dt_g} = -f\mathbf{k} \times \vec{V}_a.$$

Solution: The QG kinetic energy is defined as usual by

$$K = \frac{1}{2}\mathbf{V}_g \cdot \mathbf{V}_g = \frac{1}{2}\left(u_g^2 + v_g^2\right). \tag{33}$$

Taking $\mathbf{V}_g \cdot$ (momentum equation), gives

$$\mathbf{V}_g \cdot \frac{D\mathbf{V}_g}{Dt_g} = \frac{DK}{Dt_g} = -f\mathbf{V}_g \cdot (\mathbf{k} \times \mathbf{V}_a). \tag{34}$$

Expanding the right side gives

$$\frac{DK}{Dt_g} = f v_a u_g - f u_a v_g = -\frac{1}{\rho_0}\frac{\partial}{\partial y}(v_a p) - \frac{1}{\rho_0}\frac{\partial}{\partial x}(u_a p) + \frac{p}{\rho_0}\left(\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y}\right), \tag{35}$$

where the geostrophic wind relationships were used in the last equality. Using the QG continuity equation gives

$$\frac{DK}{Dt_g} = -\frac{1}{\rho_0}\nabla_h \cdot (u_a p, v_a p) - \frac{p}{\rho_0}\frac{\partial w}{\partial z}. \tag{36}$$

Using the hydrostatic equation on the last term yields

$$\frac{DK}{Dt_g} = -\nabla \cdot \mathbf{F}_a + \frac{g}{\overline{\theta}} w\theta, \tag{37}$$

where $\mathbf{F}_a \equiv \frac{1}{\rho_0}(u_a p,\ v_a p,\ wp)$. Due to the nondivergence of the geostrophic wind, $\mathbf{V}_g \cdot \nabla_h K = \nabla_h \cdot (\mathbf{V}_g K) \equiv \nabla_h \cdot \mathbf{F}_g$. Letting $\mathbf{S} = \mathbf{F}_a + \mathbf{F}_g$ allows (37) to be written as

$$\frac{\partial K}{\partial t} = -\nabla \cdot \mathbf{S} + \frac{g}{\overline{\theta}} w\theta, \tag{38}$$

which shows that the local change in QG kinetic energy is due to a conversion of potential energy ($w\theta$), positive for potentially cold air sinking and potentially warm air rising, and a flux divergence associated with flux vector **S**. Part of the flux relates to the geostrophic advection of K, and another part to the ageostrophic flux of pressure.

6.8. We have considered simple PV distributions associated with points and spheres. Now consider an extremely complicated PV structure, which is completely contained within a cube having sides of length L. Without knowing the detailed structure of the PV, derive a formula relating the circulation on the lateral sides of the cube, and the mean θ on the top and bottom of the cube, to the mean PV in the cube.
Hint: First show that the PV can be expressed as the divergence of a vector field.

Solution: From (6.24)

$$q - f = \frac{1}{\rho_0 f}\nabla_h^2 p + \frac{1}{\rho_0}\frac{\partial}{\partial z}\frac{f}{N^2}\frac{\partial p}{\partial z} = \nabla \cdot \mathbf{D}, \tag{39}$$

where

$$\mathbf{D} = \frac{1}{\rho_0 f}\left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{f^2}{N^2}\frac{\partial p}{\partial z}\right). \tag{40}$$

Application of the divergence theorem implies

$$\overline{q} = \frac{1}{L^3}\int_V \nabla \cdot \mathbf{D}\, dV = \frac{1}{L^3}\int_A \mathbf{D} \cdot \mathbf{n}\, dA. \tag{41}$$

Here, $\overline{q}$ is the mean QG PV in the cube, $\int_V$ denotes a volume integral, $\int_A$ indicates an integral over the surface of the volume, and $\mathbf{n}$ denotes a unit vector directed outward to the surface of the cube. The integral in (41) may be broken down into components on each face of the cube. On the top of the cube we have

$$\frac{1}{L^3}\int_A D_k\, dA \;=\; \frac{1}{L^3}\int_A \frac{f^2}{N^2}\frac{\partial p}{\partial z}\, dA = \frac{1}{L^3}\int_A \frac{d\overline{\theta}}{dz}^{-1}\theta dA = \left(\frac{d\overline{\theta}}{dz}L\right)^{-1}\overline{\theta}^{top}, \tag{42}$$

where the second equality employs the hydrostatic equation, and $\overline{\theta}^{top}$ is the potential temperature averaged over the top of the cube (Figure 6.3). Similarly, since the outward normal is in the opposite direction on the bottom of the cube, the contribution to the integral on this surface is

$$-\frac{1}{L^3}\int_A D_k\, dA \;=\; -\left(\frac{d\overline{\theta}}{dz}L\right)^{-1}\overline{\theta}^{bot}. \tag{43}$$

Along the lateral sides of the cube we use the geostrophic relaionships (see Figure 6.3). For example, on the "right" face of the cube, we have

$$\frac{1}{L^3}\int_A D_i\, dA \;=\; \frac{1}{L^3}\int_A \frac{1}{\rho_0 f}\frac{\partial p}{\partial x} = \frac{1}{L}\overline{v}_g^{right}, \tag{44}$$

where the second equality employs the geostrophic relationship and $\overline{v}_g^{right}$ is the meridional geostrophic wind averaged over the right face of the cube. Notice that this is flow along the face of the cube. Similarly, since the outward normal is in the opposite direction on the left side of the cube, the contribution to the integral is

$$-\frac{1}{L^3}\int_A D_i\,dA \;=\; -\frac{1}{L}\overline{v}_g^{left}. \tag{45}$$

On the remaining faces of the cube are contributions from $\overline{u}_g$. Taken together, the results reveal that the mean QG PV in the cube is related to the mean static stability $\left(\overline{\theta}^{top} - \overline{\theta}^{bot}\right)$ and mean circulation. That is, no matter how complicated the QG PV distribution, the averaged properties of vorticity and static stability relate to the averaged QG PV as for individual points of QG PV.

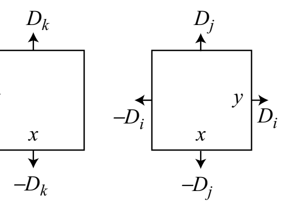

FIGURE 6.3 Faces of the cube for Problem 6.8. Arrows denote outward normal unit vectors.