# Atmospheric Oscillations

**5.1.** Show that the Fourier component $F(x) = \text{Re}[C \exp(imx)]$ can be written as $F(x) = |C| \cos m(x + x_0)$, where $x_0 = m^{-1} \sin^{-1}(C_i/|C|)$ and $C_i$ stands for the imaginary part of $C$.

**Solution:** $+F(x) = \text{Re}[C \exp(imx)] = C_r \cos mx - C_i \sin mx$, where $C = C_r + iC_i$. Define $C_r = |C| \cos mx_0,\ C_i = |C| \sin mx_0$. Thus, $mx_0 = \sin^{-1}(C_i/|C|)$, and $F(x) = |C|(\cos mx \cos mx_0 - \sin mx \sin mx_0) = |C| \cos[m(x - x_0)]$.

**5.2.** In the study of atmospheric wave motions, it is often necessary to consider the possibility of amplifying or decaying waves. In such a case we might assume that a solution has the form $\psi = A \cos(kx - \nu t - kx_0) \exp(\alpha t)$, where $A$ is the initial amplitude, $\alpha$ the amplification factor, and $x_0$ the initial phase. Show that this expression can be written more concisely as $\psi = \text{Re}\left[Be^{ik(x-ct)}\right]$, where both $B$ and $c$ are complex constants. Determine the real and imaginary parts of $B$ and $c$ in terms of $a$, $\alpha$, $k$, $\nu$, and $x_0$.

**Solution:** $\psi = \text{Re}\{B \exp[ik(x - ct)]\} = \text{Re}\{B \exp[ik(x - c_r t)]\} e^{kc_i t}$, but from Problem 7.1 we can write this as $\psi = |B| \cos[k(x - c_r t) - kx_0] e^{kc_i t} = A \cos([kx - \nu t - kx_0]) e^{\alpha t}$, where $A = |B|$, $\alpha = kc_i$, $\nu = kc_r$, and $kx_0 = -\sin^{-1}(B_i/|B|)$. Thus, $B_r = A \cos kx_0,\ B_i = -A \sin kx_0$.

**5.3.** Several of the wave types discussed in this chapter are governed by equations that are generalizations of the wave equation

$$\frac{\partial^2 \psi}{\partial t^2} = c^2 \frac{\partial^2 \psi}{\partial x^2}$$

This equation can be shown to have solutions corresponding to waves of arbitrary profile moving at the speed $c$ in both the positive and negative $x$ directions. We consider an arbitrary initial profile of the field $\psi$; $\psi = f(x)$ at $t = 0$. If the profile is translated in the positive $x$ direction at speed $c$ without change of shape, then $\psi = f(x')$, where $x'$ is a coordinate moving at speed $c$ so that $x = x' + ct$. Thus, in terms of the fixed coordinate $x$, we can write $\psi = f(x - ct)$, corresponding to a profile that moves in the positive $x$ direction at speed $c$ without change of shape. Verify that $\psi = f(x - ct)$ is a solution for any arbitrary continuous profile $f(x - ct)$.

*Hint*: Let $x - ct = x'$ and differentiate $f$ using the chain rule.

**Solution:** $\psi = f(x')$, $x' = x - ct$, $\partial x'/\partial x = 1$, $\partial x'/\partial t = -c$, Thus, by the chain rule, $\frac{\partial \psi}{\partial x} = \left(\frac{\partial f}{\partial x'}\right)\left(\frac{\partial x'}{\partial x}\right) = \frac{\partial f}{\partial x'}$. Differentiating by $x$ gives $\frac{\partial^2 \psi}{\partial x^2} = \frac{\partial^2 f}{\partial x'^2}$. Similarly, $\frac{\partial \psi}{\partial t} = \left(\frac{\partial f}{\partial x'}\right)\left(\frac{\partial x'}{\partial t}\right) = -c\frac{\partial f}{\partial x'}$, which, when differentiated by $t$, gives $\frac{\partial^2 \psi}{\partial t^2} = c^2 \frac{\partial^2 f}{\partial x'^2}$. Substituting into the wave equation then shows that $f(x')$ is indeed a solution.

**5.4.** Assuming that the pressure perturbation for a one-dimensional acoustic wave is given by (7.15), find the corresponding solutions of the zonal wind and density perturbations. Express the amplitude and phase for u′ and $\rho'$ in terms of the amplitude and phase of $p'$.

**Solution:** From the momentum equation: $\left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right) u' = -\left(\frac{1}{\overline{\rho}}\right)\frac{\partial p'}{\partial x} = -\left(\frac{ik}{\overline{\rho}}\right) Ae^{ik(x-ct)}$. Substituting in the assumed solution $u' = Be^{ik(x-ct)}$ gives $ik(c - \overline{u}) B = (ik/\overline{\rho}) A$, or $B = A[\overline{\rho}(c - \overline{u})]^{-1}$. But, $(c - \overline{u})^2 = \gamma(\overline{p}/\overline{\rho})$. Thus, $u' = p'(c - \overline{u})(\gamma \overline{p})^{-1}$. Now linearizing (5.15) yields $\left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right) \rho' = -\overline{\rho}\frac{\partial u'}{\partial x} = -ik\overline{\rho}Be^{ik(x-ct)}$. Letting $\rho' = Ce^{ik(x-ct)}$ gives $ik(c - \overline{u}) C = (ik\overline{\rho}) B$. Thus, $C = \overline{\rho}B(c - \overline{u})^{-1} = A(c - \overline{u})^{-2}$. Hence, $\rho' = p'(c - \overline{u})^{-2}$.

**5.5.** Show that for isothermal motion ($DT/Dt = 0$) the acoustic wave speed is given by $(gH)^{1/2}$, where $H = RT/g$ is the scale height.

**Solution:** Taking the log of the ideal gas law ($p = \rho RT$) gives $\ln p = \ln \rho + \ln R + \ln T$. For isothermal motion, $D \ln T/Dt = 0$, so that $D \ln p/Dt - D \ln \rho/Dt = 0$. This differs from the isentropic case (5.17) by the absence of $\gamma$. Thus, the solution is given by (7.16), with $\gamma$ replaced by unity: $c - \bar{u} = \pm (RT)^{1/2} = \pm (gH)^{1/2}$.

**5.6.** In Section 5.3.1, the linearized equations for acoustic waves were developed for the special situation of one-dimensional propagation in a horizontal tube. Although this situation does not appear to be directly applicable to the atmosphere, there is a special atmospheric mode, the *Lamb wave*, that is a horizontally propagating acoustic mode with no vertical velocity perturbation ($w' = 0$). Such oscillations have been observed following violent explosions such as volcanic eruptions and atmospheric nuclear tests. Using (5.20), (5.21), plus the linearized forms of the hydrostatic equation and the continuity equation (5.15) derives the height dependence of the perturbation fields for the Lamb mode in an isothermal basic state atmosphere assuming that the pressure perturbation at the lower boundary ($z = 0$) has the form (5.23). Determine the vertically integrated kinetic energy density per unit horizontal area for this mode.

**Solution:** When vertical velocity vanishes, the hydrostatic approximation (2.29) applies for the disturbance fields: $\partial p'/\partial z = -\rho' g$. Linearizing (5.15), and combining with (5.21) to eliminate $u'$ yields $\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right) p' - \left(\frac{\gamma \bar{p}}{\bar{\rho}}\right)\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right) \rho' = 0$, which implies that $p' = c_s^2 \rho'$. Using this result to eliminate $\rho'$ in the hydrostatic equation gives $\partial p'/\partial z = -\left(g/c_s^2\right) p'$. Thus, $p' = A \exp\left(-gz/c_s^2\right) \exp\left[ik\left(x - ct\right)\right]$. Substitution into the perturbation momentum equation (7.12) gives $\bar{\rho} u' = (c - \bar{u})^{-1} p' = (c - \bar{u})^{-1} A \exp\left(-gz/c_s^2\right) \exp\left[ik\left(x - ct\right)\right]$, where $c - \bar{u} = \pm c_s$, and $\bar{\rho} = \rho_0 \exp(-z/H)$. Averaged over a wave period, the kinetic energy density is given by $\frac{\bar{\rho} u'^2}{2} = \frac{A^2}{2\rho_0 c_s^2} \exp\left[z\left(\frac{1}{H} - \frac{2g}{c_s^2}\right)\right]$, where we have assumed that $A$ is real. But $\frac{1}{H} - \frac{2g}{c_s^2} = \frac{1}{H}\left(1 - \frac{2}{\gamma}\right) = -\left(\frac{3}{7H}\right)$, where we have used that facts that $c_s^2 = \gamma RT$, $\gamma = 7/5$, and $H = RT/g$. Thus, $\int_0^\infty \frac{\bar{\rho} u'^2}{2} dz = \frac{A^2}{2\rho_0 c_s^2}\left(\frac{7H}{3}\right) = \frac{5A^2}{6\rho_0 g}$.

**5.7.** If the surface height perturbation in a shallow water gravity wave is given by $h' = \mathrm{Re}\left[Ae^{ik(x-ct)}\right]$, find the corresponding velocity perturbation $u'(x,t)$. Sketch the phase relationship between $h'$ and $u'$ for an eastward propagating wave.

**Solution:** In rotated coordinates, from (5.37),

$$u' = \frac{\nu k g}{\nu^2 - f^2} h'$$

From (5.32) in rotated coordinates,

$$\nu^2 = g\bar{h}k^2$$

and

$$\nu = ck$$

Therefore,

$$u' = \frac{c}{\bar{h}} h'$$

Figure 5.8 shows the relationship between $u'$ and $h'$ for an eastward propagating wave.

**5.8.** Assuming that the vertical velocity perturbation for a two-dimensional internal gravity wave is given by (5.65), obtain the corresponding solution for the $u'$, $p'$, and $\theta'$ fields. Use these results to verify the approximation $\left|\rho_0\theta'/\overline{\theta}\right| \gg \left|p'/c_s^2\right|$, which was used in (5.58).

**Solution:** Since the system is linear all the disturbance fields must have solutions of similar form: $\begin{bmatrix} u \\ w' \\ p' \\ \theta' \end{bmatrix} = \mathrm{Re}\left\{\begin{bmatrix} \hat{u} \\ \hat{w} \\ \hat{p} \\ \hat{\theta} \end{bmatrix} \exp\left[i\left(kx+mz-\nu t\right)\right]\right\}$. Substitution into (5.61) then yields $\hat{u} = -\left(m/k\right)\hat{w}$, substitution into (5.62) gives $-i\left(\nu-\overline{u}k\right)\hat{\theta} = -\hat{w}d\overline{\theta}/dz$, and substitution into (5.59) gives $-i\left(\nu-\overline{u}k\right)\hat{u} = -\left(ik/\rho_0\right)\hat{p}$. The first and third of these can be combined to eliminate $\hat{u}$: $\left(\hat{p}/\rho_0\right) = -\left[m\left(\nu-\overline{u}k\right)/k^2\right]\hat{w}$. These may be checked by substitution into (5.60) and use of the dispersion relation (5.66). To verify the inequality, note that from (5.62) we can deduce (using the definition of $N^2$) that $\left(\frac{\rho_0\theta'}{\overline{\theta}}\right) = -i\rho_0\left(\nu-\overline{u}k\right)^{-1}\left(\frac{N^2}{g}\right)w'$, while from the relation between $\hat{p}$ and $\hat{w}$ given above, together with the definition $c_s^2 = \gamma RT = \gamma gH$, we obtain, $\left(\frac{p'}{c_s^2}\right) = -\left(\nu-\overline{u}k\right)\left(\frac{m\rho_0}{k^2\gamma gH}\right)w'$. Forming the required ratio and using the dispersion relation (5.66): $\left|\frac{\rho_0\theta'}{\overline{\theta}}\right| \Big/ \left|\left(\frac{p'}{c_s^2}\right)\right| = \left(\frac{N^2/g}{\upsilon-\overline{u}k}\right)\Big/\left[\frac{m(\upsilon-\overline{u}k)}{k^2\gamma gH}\right] = \gamma H\frac{(k^2+m^2)}{m}$. By inspection this ratio is large for $m \to 0$ and also for $L_z = 2\pi/m \ll H$, provided that $m^2 \gg k^2$.

**5.9.** For the situation in Problem 5.8, express the vertical flux of horizontal momentum, $\rho_0\overline{u'w'}$, in terms of the amplitude $A$ of the vertical velocity perturbation. Hence, show that the momentum flux is positive for waves in which phase speed propagates eastward and downward.

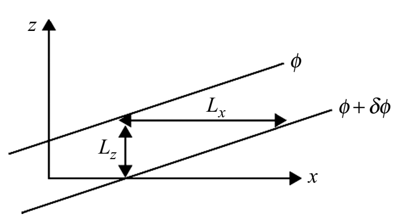
**Solution:** $\rho_0\overline{u'w'} = -\rho_0 mk^{-1}\overline{w'^2} = -\rho_0 mk^{-1}\left|A\right|^2(1/2)$, where the 1/2 comes from the fact that $\overline{\cos^2 kx} = 1/2$. Note that the momentum flux is positive for $m < 0$, which corresponds to downward phase propagation if $\left(\nu-\overline{u}k\right) > 0$.

**5.10.** Show that if (5.60) is replaced by the hydrostatic equation (i.e., the terms in $w'$ are neglected) the resulting frequency equation for internal gravity waves is just the asymptotic limit of (5.66) for waves in which $|k| \ll |m|$.

**Solution:** Neglecting $\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)w'$ in (5.60) yields in place of (5.63): $\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)\left(-\frac{\partial u'}{\partial z}\right) - \left(\frac{g}{\overline{\theta}}\right)\frac{\partial\theta'}{\partial x} = 0$. Then (5.64) becomes $\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)^2\left(\frac{\partial^2 w'}{\partial z^2}\right) + N^2\frac{\partial^2 w'}{\partial x^2} = 0$. Thus, $\left(\nu-\overline{u}k\right)^2 = N^2k^2/m^2$, or $\left(\nu-\overline{u}k\right) = \pm Nk/m$, which is the limit of (5.66) for $|k| \ll |m|$.

**5.11.** **(a)** Show that for $\overline{u} = 0$ the group velocity in two-dimensional internal gravity waves is parallel to lines of constant phase.
**(b)** Show that in the long-wave limit ($|k| \ll |m|$) the magnitudes of the zonal and vertical components of the group velocity equal the magnitudes of the zonal and vertical components of the phase speed so that energy propagates one wavelength horizontally and one wavelength vertically per wave period.

**Solution:**
**(a)** The ratio of the $x$ and $z$ components of group velocity can be evaluated from (5.67) as $\left(\frac{c_{gx}}{c_{gz}}\right) = \left(\frac{m^2}{-km}\right) = -\left(\frac{m}{k}\right) = \left(\frac{L_x}{L_z}\right)$. (Note that $m < 0$ for the situation shown in the diagram, so both components of group velocity are positive.) Since $c_{gz}/c_{gx} = L_z/L_x$, the group velocity vector relative to the mean flow has the same slope as the phase lines.



**(b)** Letting $|k| \ll |m|$, (5.67) reduces to $c_{gx} = \pm N/m$, and $c_{gz} = \mp Nk/m^2$, where for $m < 0$ and upward propagation the lower sign is taken in both formulas. But for the same conditions phase speeds in the zonal and vertical directions are $c_x = \nu/k = \pm N/m$ and $c_z = \nu/m = \pm Nk/m^2$. So zonal phase speed equals zonal group speed and the magnitude of the vertical phase speed equals the magnitude of the vertical group speed (but with opposite sign).

**5.12.** Determine the perturbation horizontal and vertical velocity fields for stationary gravity waves forced by flow over sinusoidally varying topography, given the following conditions: the height of the ground is $h = h_0 \cos kx$, where $h_0 = 50$ m is a constant; $N = 2 \times 10^{-2}\,\mathrm{s}^{-1}$; $\bar{u} = 5\,\mathrm{m\,s}^{-1}$; and $k = 3\times10^{-3}\,\mathrm{m}^{-1}$. [*Hint*: For small amplitude topography ($h_0 k \ll 1$) we can approximate the lower boundary condition by $w' = Dh/Dt = \bar{u}\partial h/\partial x$ at $z = 0$.]

**Solution:** The lower boundary condition gives $w' = -\bar{u}kh_0 \sin kx$. From (5.65) the general solution is $w' = A\cos(kx + mz) + B\sin(kx + mz)$, where from (5.64) $m^2 = N^2/\bar{u}^2 - k^2$. The lower boundary condition gives $A = 0$, and $B = -\bar{u}kh_0$. Thus, $w' = W_0 \sin(kx + mz)$, and from (5.61), $u' = U_0 \sin(kx + mz)$, with $W_0 = -\bar{u}kh_0 = 0.75\,\mathrm{m\,s}^{-1}$, and $U_0 = \bar{u}mh_0 = 0.661\,\mathrm{m\,s}^{-1}$.

**5.13.** Verify the group velocity relationship for inertia-gravity waves given in (5.88).

**Solution:** For $l = 0$ (5.87) gives $\nu^2 = f^2 + N^2k^2/m^2$, so that differentiation with respect to $k$ and $m$ gives $2\nu\frac{\partial \nu}{\partial k} = \frac{2kN^2}{m^2}$; $2\nu\frac{\partial \nu}{\partial m} = -\frac{2k^2N^2}{m^3}$. Thus, $|c_{gz}/c_{gx}| = |(\partial\nu/\partial m)/(\partial\nu/\partial k)| = |k/m| = \left(\nu^2 - f^2\right)^{1/2}/N$.

**5.14.** Show that when $\bar{u} = 0$, the wave-number vector $\boldsymbol{\kappa}$ for an internal gravity wave is perpendicular to the group velocity vector.

**Solution:** $\boldsymbol{\kappa} = (k, m)$, and $\mathbf{c}_g = \left(c_{gx}, c_{gz}\right) = Nm\left(k^2 + m^2\right)^{-3/2}(m, -k)$. Thus, $\boldsymbol{\kappa} \cdot \mathbf{c}_g = 0$, so that the wave-number and group velocity vectors are orthogonal.

**5.15.** Derive an expression for the group velocity of a barotropic Rossby wave with dispersion relation (5.110). Show that for stationary waves the group velocity always has an eastward zonal component relative to Earth. Hence, Rossby wave energy propagation must be downstream of topographic sources.

**Solution:** From (5.110) the frequency is $\nu = \bar{u}k - \beta k\left(k^2 + l^2\right)^{-1}$. Thus, $c_{gx} = \frac{\partial \nu}{\partial k} = \bar{u} - \frac{\beta}{(k^2+l^2)} + \frac{2k^2\beta}{(k^2+l^2)^2} = \bar{u} + \frac{\beta(k^2-l^2)}{(k^2+l^2)^2}$; $c_{gy} = \frac{\partial \nu}{\partial l} = +\frac{2\beta kl}{(k^2+l^2)^2}$. Now, for stationary waves $\nu = 0$, or $\bar{u} = \beta/\left(k^2 + l^2\right)$, and hence from above, $c_{gx} = \frac{\beta(k^2+l^2)}{(k^2+l^2)^2} + \frac{\beta(k^2-l^2)}{(k^2+l^2)^2} = \frac{2\beta k^2}{(k^2+l^2)^2} > 0$. (The meridional component of group velocity may be either positive or negative, depending on the sign of $l$.)

**5.16.** Prove that the internal gravity waves of Section 5.5.2 have zero linearized potential vorticity.

**Solution:** Linearized PV is given by (5.92),

$$\Pi' = \zeta' + f\frac{\partial}{\partial z}\left(\frac{\theta'}{d\bar{\theta}/dz}\right).$$

Relative vorticity for inertia-gravity waves is given by (5.94)

$$\zeta' = \frac{fm^2}{N^2}\hat{p},$$

and from the hydrostatic equation (5.79) we have

$$\theta' = \frac{\bar{\theta}}{g}im\hat{p}.$$

For constant buoyancy frequency ($N^2$),

$$f\frac{\partial}{\partial z}\left(\frac{\theta'}{d\bar{\theta}/dz}\right) = \frac{f}{d\bar{\theta}/dz}\frac{\partial\theta'}{\partial z} = -\frac{m^2 f}{N^2},$$

and substituting this, along with the relative vorticity, into the expression for the linearized PV gives

$$\Pi' = \frac{fm^2}{N^2}\hat{p} - \frac{fm^2}{N^2}\hat{p} = 0.$$