Chapter 4

Circulation, Vorticity, and Potential Vorticity

4.1. What is the circulation about a square of 1000 km on a side for an easterly (that is, westward flowing) wind that decreases in magnitude toward the north at a rate of $10\,\mathrm{m\,s^{-1}}$ per 500 km? What is the mean relative vorticity in the square?

Solution: Vorticity is: $\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 - \left(\frac{10\,\mathrm{m\,s^{-1}}}{5\times10^5\,\mathrm{m}}\right) = -2\times10^{-5}\,\mathrm{s^{-1}}$
But, $C = \iint_A \zeta\, dA = \zeta_m A = \left(-2\times10^{-5}\,\mathrm{s^{-1}}\right)\left(10^{12}\,\mathrm{m^2}\right) = -2\times10^{7}\,\mathrm{m^2\,s^{-1}}$.

4.2. A cylindrical column of air at 30°N with radius 100 km expands to twice its original radius. If the air is initially at rest, what is the mean tangential velocity at the perimeter after expansion?

Solution: From the circulation theorem $C + (2\Omega\sin\phi)\,A = $ Constant. Thus, $C_{\text{final}} = 2\Omega\sin\phi\,(A_{\text{initial}} - A_{\text{final}}) + C_{\text{initial}}$. But $A_{\text{initial}} = \pi r_i^2$, and $A_{\text{final}} = \pi r_f^2$. But $r_f = 2r_i$. Therefore, $C_{\text{final}} = 2\Omega\sin\phi\left(-3\pi r_i^2\right)$. Now $V = C_{\text{final}}\big/\left(2\pi r_f\right)$, so $V = 2\Omega\sin\phi\,(-3r_i/4) = -5.5\,\mathrm{m\,s^{-1}}$ (anticyclonic).

4.3. An air parcel at 30°N moves northward, conserving absolute vorticity. If its initial relative vorticity is $5\times10^{-5}\,\mathrm{s^{-1}}$, what is its relative vorticity upon reaching 90°N?

Solution: Absolute vorticity is conserved for the column: $\zeta + f = $ Constant. Thus, $\zeta_{\text{final}} = \zeta_{\text{initial}} + (f_{\text{initial}} - f_{\text{final}})$. Hence, $\zeta_{\text{final}} = -2\Omega\sin(\pi/2) + \left[5\times10^{-5} + 2\Omega\sin(\pi/6)\right] = -2.3\times10^{-5}\,\mathrm{s^{-1}}$.

4.4. An air column at 60°N with $\zeta = 0$ initially stretches from the surface to a fixed tropopause at 10 km height. If the air column moves until it is over a mountain barrier 2.5 km high at 45°N, what are its absolute vorticity and relative vorticity as it passes the mountaintop, assuming that the flow satisfies the barotropic potential vorticity equation?

Solution: By conservation of potential vorticity, $(\zeta + f)/H = $ Constant. Now, $\zeta = 0$ initially, so $(\zeta + f)_{\text{final}} = \left(H_{\text{final}}\big/H_{\text{initial}}\right)f_{\text{initial}} = (7.5/10.0)\left(1.263\times10^{-4}\right) = 9.473\times10^{-5}\,\mathrm{s^{-1}}$. Thus, $\zeta_{\text{final}} = 9.473\times10^{-5} - 10.312\times10^{-5} = -8.4\times10^{-6}\,\mathrm{s^{-1}}$.

4.5. Find the average vorticity within a cylindrical annulus of inner radius 200 km and outer radius 400 km if the tangential velocity distribution is given by $V = A/r$, where $A = 10^6\,\mathrm{m^2\,s^{-1}}$ and r is in meters. What is the average vorticity within the inner circle of radius 200 km?

Solution: The line integral for computing the circulation in the annular region is shown in the figure. For the outer circle $C_o = (2\pi r_o)\left(10^6/r_o\right) = 2\pi\times10^6\,\mathrm{m^2}\,s^{-1}$, and for the inner circle $C_i = (2\pi r_i)\left(-10^6/r_i\right) = -2\pi\times10^6\,\mathrm{m^2\,s^{-1}}$. (Note that C_i is taken by going clockwise around the circle, or opposite to direction of V.) Thus, since $C_o + C_i = 0$, the vorticity vanishes in the annular region. The mean vorticity in the inner cylinder is given by the circulation divided by area for the inner circle: $\zeta_m = C/A = \left(2\pi\times10^6\,\mathrm{m^2\,s^{-1}}\right)\big/\left[\pi(2\times10^5)^2\,\mathrm{m^2}\right]\left[\pi(2\times10^5)^2\,\mathrm{m^2}\right] = 5\times10^{-5}\,\mathrm{s^{-1}}$.

4.6. Show that the anomalous gradient wind cases discussed in Section 3.2.5 have negative absolute circulation in the Northern Hemisphere and hence have negative average absolute vorticity.

Solution: The absolute circulation is $C_a = 2\pi VR + f\pi R^2 = 2\pi R\left(V + fR\big/2\right)$, where R is the radius of curvature, which is negative for the anomalous cases (see Table 3.1). But from (3.15) and Table 3.1, $V + fR/2 > 0$, so $C_a < 0$, and $\bar{\zeta} + f < 0$.

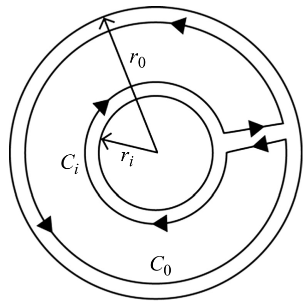

FIGURE 4.5

4.7. Compute the rate of change of circulation about a square in the (x, y) plane with corners at (0, 0), (0, L), (L, L), and (L, 0) if temperature increases eastward at a rate of 1°C per 200 km and pressure increases northward at a rate of 1 hPa per 200 km. Let $L = 1000$ km and the pressure at the point (0, 0) be 1000 hPa.

Solution: From the circulation theorem, $DC/Dt = -\oint RTd\ln p$. But $\ln p$ changes only for the North-South segments of the square, so if δT and δp are the temperature and pressure differences,

$$\frac{DC}{Dt} = -R\delta T \ln\left[\frac{(p_0 + \delta p)}{p_0}\right] = -(287)(5)(0.005) = -7.2\,\mathrm{m^2\,s^{-2}}$$

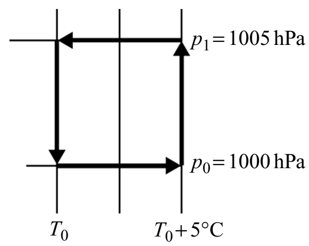

FIGURE 4.7

4.8. Verify the identity (4.14) by expanding the vectors in Cartesian components.

Solution: $(\mathbf{V}\cdot\nabla)\,\mathbf{V} = u\frac{\partial \mathbf{V}}{\partial x} + v\frac{\partial \mathbf{V}}{\partial y} = \mathbf{i}\left[u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right] + \mathbf{j}\left[u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right],$

$$\nabla\left(\frac{\mathbf{V}\cdot\mathbf{V}}{2}\right) = \nabla\left(\frac{u^2+v^2}{2}\right) = \mathbf{i}\frac{\partial}{\partial x}\left(\frac{u^2+v^2}{2}\right) + \mathbf{j}\frac{\partial}{\partial y}\left(\frac{u^2+v^2}{2}\right)$$

$$= \mathbf{i}\left[u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x}\right] + \mathbf{j}\left[u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y}\right]$$

$\mathbf{k}\times\mathbf{V}\zeta = \mathbf{k}\times(\mathbf{i}u + \mathbf{j}v)\left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right]$, but $\mathbf{k}\times\mathbf{i} = \mathbf{j}$, and $\mathbf{k}\times\mathbf{j} = -\mathbf{i}$. Thus,
$\mathbf{k}\times\mathbf{V}\zeta = \mathbf{j}\left[u\frac{\partial v}{\partial x} - u\frac{\partial u}{\partial y}\right] - \mathbf{i}\left[v\frac{\partial v}{\partial x} - v\frac{\partial u}{\partial y}\right]$, and by inspection
$\nabla\left(\frac{\mathbf{V}\cdot\mathbf{V}}{2}\right) + \mathbf{k}\times\mathbf{V}\zeta = (\mathbf{V}\cdot\nabla)\,\mathbf{V}$, which was to be proved.

4.9. Derive a formula for the dependence of depth on radius for an incompressible fluid in solid-body rotation in a cylindrical tank with a flat bottom and free surface at the upper boundary. Let H be the depth at the center of the tank, Ω the angular rotation rate of the tank, and a the radius of the tank.

Solution: Centrifugal force must balance the pressure gradient force. Let the depth be h, then $\Omega^2 r = g\left(\partial h/\partial r\right)$. Thus, $dh = \Omega^2 r dr/g$. Integrating from the center outward gives $\int_H^{h(r)} dh = \int_o^r \left(\Omega^2 r/g\right) dr$, which yields $h(r) = H + \left(\Omega^2 r^2/2g\right)$.

4.10. By how much does the relative vorticity change for a column of fluid in a rotating cylinder if the column is moved from the center of the tank to a distance 50 cm from the center? The tank is rotating at the rate of 20 revolutions per minute, the depth of the fluid at the center is 10 cm, and the fluid is initially in solid-body rotation.

Solution: Potential vorticity is conserved: $\left(\frac{\zeta_0+2\Omega}{H_0}\right) = \left(\frac{\zeta_1+2\Omega}{H_1}\right)$ with $H_0 = 10\,\text{cm}$ and $H_1 = H_0 + \Omega^2 r^2 / 2g$ (from Problem 4.9). But, $\zeta_0 = 0$ and $\Omega = 2.09$ rad/s. Thus, $\zeta_1 = \left(\frac{H_1-H_0}{H_0}\right) 2\Omega = \frac{(5.57)(4.18)}{(10)} = 2.3\,\text{s}^{-1}$.

4.11. A cyclonic vortex is in cyclostrophic balance with a tangential velocity profile given by the expression $V = V_0(r/r_0)^n$, where V_0 is the tangential velocity component at the distance r_0 from the vortex center. Compute the circulation about a streamline at radius r, the vorticity at radius r, and the pressure at radius r. (Let p_0 be the pressure at r_0 and assume that density is a constant.)

Solution: $C = 2\pi V r = 2\pi V_0 r^{n+1} / r_0^n$, $\zeta = \frac{\partial V}{\partial r} + \frac{V}{r} = \frac{1}{r}\frac{\partial (rV)}{\partial r} = \frac{V_0}{r_0^n r}\frac{\partial \left(r^{n+1}\right)}{\partial r} = (n+1)\frac{V_0}{r_0^n} r^{n-1} = (n+1)\frac{V}{r}$, Now, cyclostrophic balance implies $\frac{V^2}{r} = \frac{1}{\rho}\frac{\partial p}{\partial r}$, which may be integrated to give $\int_{p_0}^{p} \frac{dp}{\rho} = \int \frac{V_0^2 r^{2n-1}}{r_0^{2n}} dr$, so that $\frac{p-p_0}{\rho} = \frac{V_0^2}{r_0^{2n}}\left(\frac{r^{2n}-r_0^{2n}}{2n}\right) = \frac{V^2-V_0^2}{2n}$.

4.12. A westerly zonal flow at 45° is forced to rise adiabatically over a north-south–oriented mountain barrier. Before striking the mountain, the westerly wind increases linearly toward the south at a rate of $10\,\text{m s}^{-1}$per 1000 km. The crest of the mountain range is at 800-hPa, and the tropopause, located at 300 hPa, remains undisturbed. What is the initial relative vorticity of the air? What is its relative vorticity when it reaches the crest if it is deflected 5° latitude toward the south during the forced ascent? If the current assumes a uniform speed of $20\,\text{m s}^{-1}$during its ascent to the crest, what is the radius of curvature of the streamlines at the crest?

Solution: By potential vorticity conservation $\left(\frac{f_0+\zeta_0}{\delta p_0}\right) = \left(\frac{f_1+\zeta_1}{\delta p_1}\right)$. But, $\zeta_0 = -\partial u / \partial y = 10^{-5}\,\text{s}^{-1}$, $f_0 = 2\Omega \sin(\pi/4) = 1.03 \times 10^{-4}\,\text{s}^{-1}$, $\delta p_0 = 700\,\text{hPa}$, $f_1 = 2\Omega \sin(2\pi/9) = 9.37 \times 10^{-5}\,\text{s}^{-1}$, $\delta p_1 = 500\,\text{hPa}$. Thus, $\zeta_1 = -1.29 \times 10^{-5}\,\text{s}^{-1}$. For a uniform flow the vorticity is entirely due to curvature so that $\zeta_1 = V/R$. Thus, $R = V/\zeta_1 = -1.55 \times 10^6\,\text{m}$ (–1550 km).

4.13. A cylindrical vessel of radius a and constant depth H rotating at an angular velocity Ω about its vertical axis of symmetry is filled with a homogeneous, incompressible fluid that is initially at rest with respect to the vessel. A volume of fluid V is then withdrawn through a point sink at the center of the cylinder, thus creating a vortex. Neglecting friction, derive an expression for the resulting relative azimuthal velocity as a function of radius (i. e., the velocity in a coordinate system rotating with the tank). Assume that the motion is independent of depth and that $V \ll \pi a^2 H$. Also compute the relative vorticity and the relative circulation.

Solution: For $V \ll \pi a^2 H$, we can neglect change in depth when considering the change, δr, in the radial position of the fluid parcels. The radial displacement δr is just that for which the volume of fluid contained between the radius r and $r - \delta r$ is equal to V: Thus, $2\pi r \delta r H = V$. But by conservation of angular momentum, $\Omega r^2 = \Omega (r-\delta r)^2 + u(r-\delta r)$, where u is the azimuthal velocity acquired during the displacement δr. Thus, $u(r-\delta r) \approx -2\Omega r \delta r = \Omega V / (\pi H)$, so that $u \approx \Omega V / (\pi r H) \propto r^{-1}$. Now, $\zeta = \frac{\partial u}{\partial r} + \frac{u}{r} = 0$, except at the origin where $\zeta \to \infty$. But the circulation is a constant independent of r: $C = 2\pi r u = 2\Omega V / H$.

4.14. (a) How far must a zonal ring of air initially at rest with respect to Earth's surface at 60° latitude and 100-km height be displaced latitudinally in order to acquire an easterly (east to west) component of 10 m s^{-1} with respect to Earth's surface? (b) To what height must it be displaced vertically in order to acquire the same velocity? Assume a frictionless atmosphere.

Solution: An approximate solution can be obtained by considering conservation of angular momentum for small changes δR in the distance to the axis of rotation of Earth: $\Omega R^2 = \left[\Omega + \frac{\delta u}{(R+\delta R)}\right](R+\delta R)^2$, where $\delta u = 10\,\text{ms}^{-1}$, the change in zonal wind under angular momentum conservation. Expanding gives $\delta u \approx -2\Omega \delta R$. Now, for latitudinal displacement

$\delta R = -\delta y \sin\phi$, while for vertical displacement $\delta R = \delta z \cos\phi$. Thus, for 60° latitude we find $\delta y = -79\,\text{km}$, or 0.71° equatorward, and $\delta z = 138\,\text{km}$.

4.15. The horizontal motion within a cylindrical annulus with permeable walls of inner radius 10 cm, outer radius 20 cm, and 10-cm depth is independent of height and azimuth and is represented by the expressions $u = 7 - 0.2r$, $v = 40 + 2r$, where u and v are the radial and tangential velocity components in cm s^{-1}, positive outward and counterclockwise, respectively, and r is distance from the center of the annulus in cm. Assuming an incompressible fluid, find (a) the circulation about the annular ring, (b) the average vorticity within the annular ring, (c) the average divergence within the annular ring, and (d) the average vertical velocity at the top of the annulus if it is zero at the base.

Solution: From Problem 4.5: $C = 2\pi r_0 v(r_0) - 2\pi r_1 v(r_1) = 2\pi\,[(20)(80) - (10)(60)]$.

a. $C = 6280\,\text{cm}^2\,\text{s}^{-1}$;

b. $\zeta = C/A = 6280/\left(\pi r_0^2 - \pi r_1^2\right) = 6.67\,\text{s}^{-1}$.

c. To calculate the divergence note that from the divergence theorem $\iint \nabla\cdot\mathbf{V}dA = A\left(\nabla\cdot\mathbf{V}\right)_{\text{mean}} = \oint_{r_0}\mathbf{V}\cdot\mathbf{n}ds - \oint_{r_1}\mathbf{V}\cdot\mathbf{n}ds$, and $A = \pi\left(r_0^2 - r_1^2\right)$, so that $\left(\nabla\cdot\mathbf{V}\right)_{\text{mean}} = \left[2\pi r_0 u(r_0) - 2\pi r_1 u(r_1)\right]/\left[\pi\left(r_0^2 - r_1^2\right)\right] = 0.0667\,\text{s}^{-1}$.

d. Now from the continuity equation, $\left(\partial w/\partial z\right)_{\text{mean}} = -\left(\nabla\cdot\mathbf{V}\right)_{\text{mean}}$, so that integrating in height gives: $w_{\text{mean}} = -H\left(\nabla\cdot\mathbf{V}\right)_{\text{mean}} = -0.667\,\text{cm}\,\text{s}^{-1}$ at $z = H$.

4.16. Prove that, as stated below Eq. (4.52), the globally averaged isentropic vorticity on an isentropic surface that does not intersect the ground must be zero. Show that the same result holds for the isobaric vorticity on an isobaric surface.

Solution: From the vector identity above (4.52) $\zeta_\theta = \mathbf{k}\cdot\left(\nabla_\theta\times\mathbf{V}\right) = \nabla_\theta\cdot\left(\mathbf{V}\times\mathbf{k}\right)$. Thus, $\int\zeta_\theta dA = \int\nabla_\theta\cdot\left(\mathbf{V}\times\mathbf{k}\right)dA = 0$, since the divergence of any vector vanishes when integrated over the sphere. The same result holds for isobaric coordinates if the subscript θ is replaced by p in all the expressions above.