Basic Conservation Laws

2.1. A ship is steaming northward at a rate of $10\,\mathrm{km\,h^{-1}}$. The surface pressure increases toward the northwest at a rate of $5\ \mathrm{Pa\ km^{-1}}$. What is the pressure tendency recorded at a nearby island station if the pressure aboard the ship decreases at a rate of 100 Pa/3 h?

Solution: $\frac{\partial p}{\partial t} = \frac{Dp}{Dt} - \mathbf{V}\cdot\nabla p$. But, $\mathbf{V}\cdot\nabla p = |\mathbf{V}|\,|\nabla p|\cos\alpha$, where α is the angle between the velocity and pressure gradient vectors (45° here). $\frac{\partial p}{\partial t} = -\left(\frac{100}{3}\ \mathrm{Pa/h}\right) - (10\ \mathrm{km/h})\ (5\ \mathrm{Pa/km})\left(1/\sqrt{2}\right) = -68.7\ \mathrm{Pa/h}$, or $\cong -2\ \mathrm{hPa/3h}$.

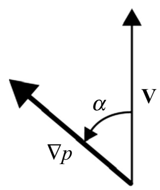

FIGURE 2.1

2.2. The temperature at a point 50 km north of a station is 3°C cooler than at the station. If the wind is blowing from the northeast at $20\ \mathrm{m\ s^{-1}}$ and the air is being heated by radiation at the rate of $1^\circ\mathrm{C\,h^{-1}}$, what is the local temperature change at the station?

Solution: $\frac{\partial T}{\partial t} = \frac{DT}{Dt} - \mathbf{V}\cdot\nabla T,$ where here $\frac{DT}{Dt} = J = 1^\circ\mathrm{C\,h^{-1}}$. Now, $\mathbf{V}\cdot\nabla T = \left(20\,\mathrm{m\,s^{-1}}\right)\left(3^\circ\mathrm{C}/5\times10^4\ \mathrm{m}\right)\left(1/\sqrt{2}\right) = 8.47\times10^{-4}\,{}^\circ\mathrm{C\,s^{-1}}$ or $3.05^\circ\mathrm{C\ h^{-1}}$. Thus, $\partial T/\partial t = 1^\circ\mathrm{C/h} - 3.05^\circ\mathrm{C/h} = -2.05^\circ\mathrm{C/h}$.

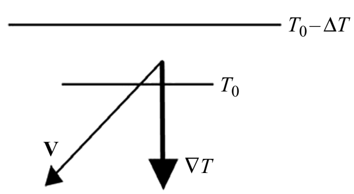

FIGURE 2.2

2.3. Derive the relationship $\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r}) = -\Omega^2\,\mathbf{R}$, which was used in Eq. (2.7).

Solution: $|\mathbf{R}| = |\mathbf{r}|\sin\alpha$, where α is the angle between $\boldsymbol{\Omega}$ and $\mathbf{r}$, and $\mathbf{R}$ is perpendicular to $\boldsymbol{\Omega}$. Then by definition of the cross product $\boldsymbol{\Omega}\times\mathbf{r} = \boldsymbol{\Omega}\times\mathbf{R} = \Omega\,|\mathbf{R}|\,\mathbf{n}$, where $\mathbf{n}$ is the unit vector pointing into the paper (see Fig. 2.3) perpendicular to the plane of $\boldsymbol{\Omega}$ and $\mathbf{R}$. Then $\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r}) = \boldsymbol{\Omega}\times\mathbf{n}\Omega\,|\mathbf{R}|$. But $\boldsymbol{\Omega}\times\mathbf{n} = -\Omega\mathbf{R}/|\mathbf{R}|$, which by substitution gives the desired result.

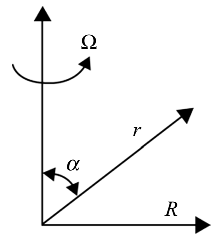

FIGURE 2.3

2.4. Derive the expression given in Eq. (2.13) for the rate of change of **k** following the motion.

Solution: $D\mathbf{k}/Dt = u\partial\mathbf{k}/\partial x + v\partial\mathbf{k}/\partial y$. From the figure, $\left|\frac{\partial \mathbf{k}}{\partial y}\right| = \lim_{\delta y \to 0}\left(\left|\frac{\delta \mathbf{k}}{\delta y}\right|\right)$, but by similarity of triangles, $|\delta\mathbf{k}|/|\mathbf{k}| = |\delta\mathbf{k}| = \delta y/a$, and $\delta\mathbf{k}$ is parallel to **j**, so that $\partial\mathbf{k}/\partial y = \mathbf{j}/a$. By similar arguments $\partial\mathbf{k}/\partial x = \mathbf{i}/a$. Thus, $D\mathbf{k}/Dt = \mathbf{i}u/a + \mathbf{j}v/a$.

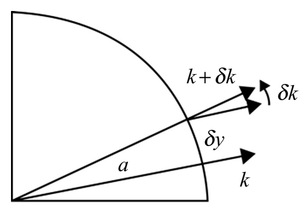

FIGURE 2.4

2.5. Suppose a 1-kg parcel of dry air is rising at a constant vertical velocity. If the parcel is being heated by radiation at the rate of $10^{-1}\,\mathrm{W\,kg^{-1}}$, what must the speed of rise be to maintain the parcel at a constant temperature?

Solution: From (2.42) $c_p DT/Dt - \alpha Dp/Dt = J$, but from the hydrostatic equation $\alpha Dp = -gDz$, Thus, $c_p DT/Dt + gDz/Dt = c_p DT/Dt + gw = J$, so that for constant parcel temperature $(DT/Dt = 0)$, $w = J/g = \left(10^{-1}\mathrm{W\,kg^{-1}}\right)/\left(9.8\,\mathrm{m\,s^{-2}}\right) = 0.0102\,\mathrm{m\,s^{-1}}$ or about $1.0\,\mathrm{cm\,s^{-1}}$.

2.6. Derive an expression for the density ρ that results when an air parcel initially at pressure p_s and density ρ_s expands adiabatically to pressure p.

Solution: For adiabatic expansion potential temperature is conserved. Thus, $\theta = T_s = T\,(p_s/p)^{R/c_p}$. Substituting from the ideal gas law, $T = p/(\rho R)$ gives $p_s/\rho_s = (p/\rho)\,(p_s/p)^{R/c_p}$ or $\rho = \rho_s\,(p/p_s)^{c_v/c_p}$, where we have used $c_v = c_p - R$.

2.7. An air parcel that has a temperature of 20°C at the 1000 hPa level is lifted dry adiabatically. What is its density when it reaches the 500 hPa level?

Solution: From Problem 2.6, $\rho = (p_s/RT_s)\,(p/p_s)^{c_v/c_p}$ so that

$$\rho(500\,\mathrm{hPa}) = \left[\frac{10^5\,\mathrm{Pa}}{\left(287\,\mathrm{J\,K^{-1}\,kg^{-1}}\right)(293\,\mathrm{K})}\right]\left(\frac{1}{2}\right)^{0.714} = 0.725\,\mathrm{kg\,m^{-3}}.$$

2.8. Suppose an air parcel starts from rest at the 800 hPa level and rises vertically to 500 hPa while maintaining a constant 1°C temperature excess over the environment. Assuming that the mean temperature of the 800–500 hPa layer is 260 K, compute the energy released owing to the work of the buoyancy force. Assuming that all the released energy is realized as kinetic energy of the parcel, what will the vertical velocity of the parcel be at 500 hPa?

Solution: Substituting from the ideal gas law $\rho = p/RT$ into (2.51), we get $\frac{Dw}{Dt} = g\left(\frac{T_0^{-1} - T^{-1}}{T^{-1}}\right) = g\left(\frac{T - T_0}{T_0}\right)$, which gives the force per unit mass. But the energy released is given by force × distance traveled. From the hypsometric equation (1.30),

$\delta Z = (RT_0/g)\ln(p_1/p_2)$. Thus, energy released per unit mass $= g\delta Z\,[(T - T_0)/T_0] = R(T - T_0)\ln(p_1/p_2) = 135\,\mathrm{J\,kg^{-1}}$. If all goes into kinetic energy, then $w^2/2 = 135\,\mathrm{m^2\,s^{-2}}$, so that $w = 16.43\,\mathrm{m\,s^{-1}}$ at 500 hPa.

2.9. Show that for an atmosphere with an adiabatic lapse rate (i.e., constant potential temperature) the geopotential height is given by $Z = H_\theta\left[1 - \left(\frac{p}{p_0}\right)^{R/c_p}\right]$, where p_0 is the pressure at $Z = 0$ and $H_\theta \equiv c_p\theta/g_0$ is the total geopotential height of the atmosphere.

Solution: In Problem 1.16, let $\gamma = g/c_p$ (adiabatic lapse rate), and note that for $p_0 = 1000\,\mathrm{hPa}$ $T_0 = \theta$. Then substitution into the formula of Problem 1.16 immediately yields the above result.

2.10. In the *isentropic* coordinate system (see Section 4.6) potential temperature is used as the vertical coordinate. Since in adiabatic flow potential temperature is conserved following the motion, isentropic coordinates are useful for tracing the actual paths of travel of individual air parcels. Show that the transformation of the horizontal pressure gradient force from z to θ coordinates is given by

$$\frac{1}{\rho}\nabla_z p = \nabla_\theta M,$$

where $M \equiv c_pT + \Phi$ is the Montgomery streamfunction.

Solution: In (1.36) let $s = \theta$. Then: $\left(\frac{\partial p}{\partial x}\right)_\theta = \left(\frac{\partial p}{\partial z}\right)\left(\frac{\partial z}{\partial x}\right)_\theta + \left(\frac{\partial p}{\partial x}\right)_z$. Letting $\frac{\partial p}{\partial z} = -\rho g$ then gives $\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right)_z = \frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right)_\theta + \left(\frac{\partial \Phi}{\partial x}\right)_\theta$, where $g dz = d\Phi$. Now with the aid of the ideal gas law $\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right)_\theta = RT\left(\frac{\partial \ln p}{\partial x}\right)_\theta$. But from (2.44): $\ln\theta = \ln T - \left(R/c_p\right)\ln p + \text{Constant}$, from which $\left(\frac{\partial \ln p}{\partial x}\right)_\theta = \frac{c_p}{R}\left(\frac{\partial \ln T}{\partial x}\right)_\theta$. Thus, $\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right)_z = c_pT\left(\frac{\partial \ln T}{\partial x}\right)_\theta + \left(\frac{\partial \Phi}{\partial x}\right)_\theta$ which simplifies to $\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right)_z = \left(\frac{\partial M}{\partial x}\right)_\theta$.

The other components can be derived in an analogous fashion.

2.11. French scientists have developed a high-altitude balloon that remains approximately at constant potential temperature as it circles Earth. Suppose such a balloon is in the lower equatorial stratosphere where the temperature is isothermal at 200 K. If the balloon were displaced vertically from its equilibrium level by a small distance δz, it would tend to oscillate about the equilibrium level. What is the period of this oscillation?

Solution: From Eq. (2.52), the parcel oscillation frequency is $N = \left[g\frac{d\ln\theta_0}{dz}\right]^{1/2}$. But for isothermal conditions from (2.47), we obtain $d\ln\theta_0/dz = g/\left(c_pT\right) = (9.8)/[(1003)\,(200)] = 4.885 \times 10^{-5}\ \mathrm{m^{-1}}$. Thus, the period of the oscillation is $2\pi/N = 2\pi/\left[(9.8)\left(4.885 \times 10^{-5}\right)\right]^{1/2} = 287\,\mathrm{s}$, or about 4.8 minutes.

2.12. Derive the approximate thermodynamic energy equation (2.55) using the scaling arguments of Sections 2.4 and 2.7.

Solution: Separating temperature and pressure, respectively, into standard atmosphere components (dependent only on z) and deviations yields the expressions $T_{\text{tot}} = T_0(z) + T(x, y, z, t) = T_0(1 + T/T_0)$, and $p_{\text{tot}} = p_0(z) + p(x, y, z, t) = p_0(1 + p/p_0)$, where T_{tot} designates the total temperature field and p_{tot} the total pressure field. Then since $|T/T_0| << 1$ and $|p/p_0| << 1$, $\ln T_{\text{tot}} \approx \ln T_0 + T/T_0$, and $\ln p_{\text{tot}} \approx \ln p_0 + p/p_0$, and (2.43) becomes approximately

$$c_pD\ln T_0\big/Dt + \left(c_p\big/T_0\right)DT\big/Dt - RD\ln p_0\big/Dt - \left(R\big/p_0\right)Dp\big/Dt = J\big/T_0.$$

But since T_0 and p_0 depend only on z, the above can be rewritten with the aid of the ideal gas law as

$$\frac{DT}{Dt} - \frac{1}{c_p\rho_0}\frac{Dp}{Dt} + w\left(\frac{dT_0}{dz} - \frac{1}{c_p\rho_0}\frac{dp_0}{dz}\right) = \frac{J}{c_p},$$

and application of the hydrostatic equation $\partial p_0/\partial z = -\rho_0 g$ gives for synoptic-scale motions (in which vertical advection of T and p is small compared with horizontal advection)

$$\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) - \left(\frac{1}{c_p\rho_0}\right)\left(\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y}\right) + w\left(\frac{g}{c_p} + \frac{dT_0}{dz}\right) \approx \frac{J}{c_p}.$$

Using the definition of the adiabatic lapse rate (2.48) and recalling that for synoptic-scale systems $\delta T \sim 4°\mathrm{C}$, $J/c_p \leq 1°\mathrm{C}$, and $\delta p/\rho_0 \sim 10^3\ \mathrm{m^2\,s^{-2}}$, the dominant terms are

$$\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) + w\left(\Gamma_d - \Gamma\right) \approx 0.$$