Introduction

1.1. Neglecting the latitudinal variation in the radius of Earth, calculate the angle between the gravitational force and gravity vectors at the surface of Earth as a function of latitude. What is the maximum value of this angle?

Solution: Let ϕ be latitude and α the angle between **g*** and **g.** (See Figure 1.5 in the text.) Then from the law of sines:

$$|\mathbf{g}|/\sin\phi = \Omega^2\,|\mathbf{R}|/\sin\alpha$$

But $|\mathbf{R}| = a\cos\phi$, where a is the radius of Earth. Thus,

$$\sin\alpha \approx \alpha = \frac{\Omega^2 a\sin\phi\cos\phi}{|\mathbf{g}|} = \frac{\Omega^2 a\sin 2\phi}{2g}$$

The maximum, which occurs at 45°, is $\alpha = 1.73\times 10^{-3}$ rad, or 0.099°.

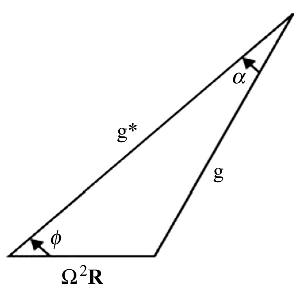

FIGURE 1.1

1.2. Calculate the altitude at which an artificial satellite orbiting in the equatorial plane can be a synchronous satellite (i.e., can remain above the same spot on the surface of Earth).

Solution: From text Eq. (1.7)

$$\mathbf{g}^* + \Omega^2\mathbf{R} = 0,\ \text{but}\ \mathbf{g}^* = -\left(GM/r^2\right)(\mathbf{r}r)$$

and at the equator $\mathbf{R} = \mathbf{r}$. Thus,

$$\Omega^2 r = \left(GM/r^2\right)\ \text{so that}\ r = \left(GM/\Omega^2\right)^{13} \approx 42{,}200\,\text{km}.$$

Then $z = r - a \approx 35{,}830\,\text{km}$.

1.3. An artificial satellite is placed into a natural synchronous orbit above the equator and is attached to Earth below by a wire. A second satellite is attached to the first by a wire of the same length and is placed in orbit directly above the first at the same angular velocity. Assuming that the wires have zero mass, calculate the tension in the wires per unit mass of satellite. Could this tension be used to lift objects into orbit with no additional expenditure of energy?

Solution: Tension per unit mass, T, is given by

$$T = \Omega^2 r_1 - GM/r_1^2,$$

where r_1 is the distance of the outer satellite from the center of Earth. Now from Problem 1.2 $r_1 = 42{,}200\,\text{km} + 35{,}830\,\text{km} \approx 78{,}000\,\text{km}$, so $T = 0.349\,\text{N}\,\text{kg}^{-1}$. This tension could be used once to lift a mass less than $T/g \approx 3.56\%$ of the mass of the outer satellite into orbit, but only by displacing the satellites and wire further into space. ($m_w g = Tm_s$, where m_w and m_s are the mass to be lifted and the mass of the outer satellite, respectively.)

1.4. A train is running smoothly along a curved track at the rate of $50\,\text{m}\,\text{s}^{-1}$. A passenger standing on a set of scales observes that his weight is 10% greater than when the train is at rest. The track is banked so that the force acting on the passenger is normal to the floor of the train. What is the radius of curvature of the track?

Solution: The weight measured by the scales (1.1 mg) is the vector sum of the forces of gravity (mg) and the centrifugal force owing to the curvature of the tracks (mu^2/r, where m is the mass of the train, u is the speed and r the radius of curvature Thus,

$$(1.1\,mg)^2 = (mg)^2 + \left(mu^2/r\right)^2.$$

Thus, $r = u^2/\left(g\sqrt{0.21}\right) = 556\,\text{m}$.

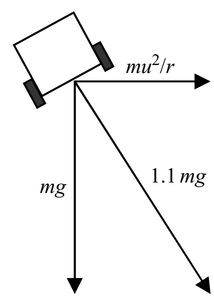

FIGURE 1.4

1.5. If a baseball player throws a ball a horizontal distance of 100 m at 30° latitude in 4 s, by how much is it deflected laterally as a result of the rotation of Earth?

Solution: Assume that the ball is thrown eastward and that $u = x/t$ is constant, where $x = 100\,\text{m}$ and $t = 4\,\text{s}$. From (1.21) $v = Dy/Dt = -\,(2\Omega \sin\phi)\,ut$, but $\sin\phi = 0.5$. Thus, in this case, $v = -\Omega ut$, integrating in time yields

$$y = -\Omega ut^2/2 = -\Omega xt/2 = -\left(7.292 \times 10^{-5}\right)(100 \times 4)/2\,\text{m}, \text{ or } y = -1.46\,\text{cm}.$$

Thus, the ball is deflected to the right.

1.6. Two balls 4 cm in diameter are placed 100 m apart on a frictionless horizontal plane at 43°N. If the balls are impulsively propelled directly at each other with equal speeds, at what speed must they travel so that they just miss each other?

Solution: From Problem 1.5 the lateral deflection for each ball is to the right and given by $|y| = (2\Omega \sin\phi)\,xt/2 = 2\,\text{cm}$ for the balls to just miss each other. Solving for t:

$$t = \frac{2\,|y|}{(2\Omega \sin\phi)\,x} = \frac{(2)\left(2 \times 10^{-2}\,\text{m}\right)}{\left(10^{-4}\,\text{s}^{-1}\right)(50\,\text{m})} = 8\,\text{s}; \text{ thus, } u = \frac{x}{t} = \frac{(50\,\text{m})}{(8\,\text{s})} = 6.25\,\text{m}\,\text{s}^{-1}.$$

1.7. A locomotive of 2×10^5 kg mass travels $50\,\text{m}\,\text{s}^{-1}$ along a straight horizontal track at 43°N. What lateral force is exerted on the rails? Compare the magnitudes of the upward reaction force exerted by the rails for cases where the locomotive is traveling eastward and westward, respectively.

Solution: The lateral force on the rails is just the reaction to the deflection force due to the Coriolis effect. Thus, for a locomotive of mass M,

$$F = -fMu = -\left(10^{-4}\,\mathrm{s}^{-1}\right)\left(2 \times 10^{5}\,\mathrm{kg}\right)\left(50\,\mathrm{m\,s}^{-1}\right) = -10^{3}\,\mathrm{N}$$

(directed southward for a train moving eastward in the Northern Hemisphere). The upward reaction force is just the weight minus the vertical component of the Coriolis force (1.18). Thus, $F_{\text{upward}} = M\,(g - 2\Omega\cos\phi u)$, and the force difference between westward and eastward travel is

$$\delta F = F_{\text{westward}} - F_{\text{eastward}} = 4\Omega\cos\phi\,|u|\,M = 2 \times 10^{3}\,\mathrm{N}.$$

1.8. Find the horizontal displacement of a body dropped from a fixed platform at a height h at the equator neglecting the effects of air resistance. What is the numerical value of the displacement for $h = 5\,\mathrm{km}$?

Solution: From text Eq. (1.16) $(Du/Dt)_{\mathrm{Co}} = -2\Omega w$ at the equator. But $Dw/Dt \approx -g$, so that $w = Dz/Dt \approx -gt$ and integration in time gives $z = gt^2/2$. Thus, $t_0 = (2h/g)^{1/2}$ is the total time of the fall from height h. From these we get $Du/Dt = 2\Omega gt$, which integrates to give $u = Dx/Dt = \Omega gt^2$. Integrating again, we obtain

$$x = \Omega gt^3/3 = (\Omega g/3)\,(2h/g)^{3/2} = 7.8\,\mathrm{m}\text{ eastward for } h = 5\,\mathrm{km}.$$

1.9. A bullet is fired directly upward with initial speed w_0, at latitude ϕ. Neglecting air resistance, by what distance will it be displaced horizontally when it returns to the ground? (Neglect $2\Omega u\cos\phi$ compared to g in the vertical momentum equation.)

Solution: $Dw/Dt = -g$ so that $w = w_0 - gt = 0$ at the top of the trajectory. Total time of flight is twice the time to the top, or $t_0 = 2w_0/g$. From text Eq. (1.16) $(Du/Dt) = -\,(2\Omega\cos\phi)\,w$, which integrates to give

$$u = -2\Omega\cos\phi\left(w_0 t - gt^2/2\right)$$

and since

$$Dx/Dt = u, x = -2\Omega\cos\phi\left(w_0 t^2/2 - gt^3/6\right).$$

Substituting the total time t_0 then gives $x = -2\Omega\cos\phi\left(2w_0^3/3g^2\right)$.

1.10. A block of mass $M = 1\,\mathrm{kg}$ is suspended from the end of a weightless string. The other end of the string is passed through a small hole in a horizontal platform and a ball of mass $m = 10\,\mathrm{kg}$ is attached. At what angular velocity must the ball rotate on the horizontal platform to balance the weight of the block if the horizontal distance of the ball from the hole is 1 m? While the ball is rotating, the block is pulled down 10 cm. What is the new angular velocity of the ball? How much work is done in pulling down the block?

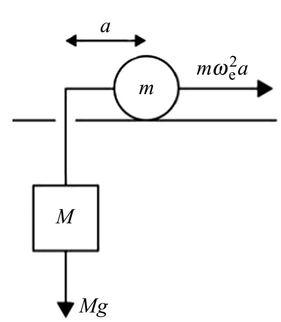

FIGURE 1.10

Solution: Force balance for equilibrium (see Figure 1.10) is $Mg = m\omega_e^2 a_e$, where ω_e is the angular velocity of the ball and $a_e = 1$ m. Solving for ω_e yields

$$\omega_e = (Mg/ma_e)^{1/2} \approx 1\ \text{s}.$$

If the block is pulled down 10 cm, then the distance of the ball from the hole is 0.9 m. By conservation of angular momentum, $\omega_f a_f^2 = \omega_e a_e^2$, where subscript f denotes the final state. The final angular velocity then is

$$\omega_f = \omega_e \left(a_e^2/a_f^2\right) = 1.222\ \text{s}^{-1}.$$

The total work done is given by the sum of the changes in kinetic and potential energy: $\delta W = \delta K + \delta P$. Now

$$\delta K = (m/2)\left[\left(\omega_f a_f\right)^2 - (\omega_e a_e)^2\right] = 1.15\ \text{J};\ \ \delta P = Mg\delta h,$$

where $\delta h = -10$ cm height change. Thus, $\delta P = -0.98$ J and $\delta W = 1.15 - 0.98 = 0.17$ J.

1.11. A particle is free to slide on a horizontal frictionless plane located at a latitude ϕ on Earth. Find the equation governing the path of the particle if it is given an impulsive northward velocity $v = V_0$ at $t = 0$. Give the solution for the position of the particle as a function of time. (Assume that the latitudinal excursion is sufficiently small so that f is constant.)

Solution: Combining text Eqs. (1.17) and (1.18), we obtain an equation for v: $D^2v/Dt^2 = -f^2 v$. The general solution is $v = A\cos ft + B\sin ft$. The initial conditions then give $A = v_0$, $B = 0$. Therefore, $v = Dy/Dt = v_0 \cos ft$, and from (1.17) we find that $u = Dx/Dt = v_0 \sin ft$. Letting (x_0, y_0) be the initial position and integrating with respect to time then yields

$$x = x_0 - (v_0/f)(\cos ft - 1), \quad y = y_0 + (v_0/f)\sin ft.$$

1.12. Calculate the 1000–500 hPa thickness for isothermal conditions with temperatures of 273 K and 250 K, respectively.

Solution: From text Eq. (1.23),

$$Z_T = (R\langle T\rangle/g_0)\ln(p_1/p_2).$$

Thus, for temperature of 273 K, $Z_T = [(287)(273)/9.8]\ln 2 = 5536$ m, while for $T = 250$ K, $Z_T = 5070$ m.

1.13. Isolines of 1000–500 hPa thickness are drawn on a weather map using a contour interval of 60 m. What is the corresponding layer mean temperature interval?

Solution: From (1.23) $\delta Z_T = (R\langle\delta T\rangle/g_0)\ln(p_1/p_2)$. Thus,

$$\delta\langle T\rangle = [g_0\delta Z_T/R\ln(p_1/p_2)] = [(9.8)(60)/(287)(\ln 2)] = 2.96 \approx 3^\circ\text{C}.$$

1.14. Show that a homogeneous atmosphere (density independent of height) has a finite height that depends only on the temperature at the lower boundary. Compute the height of a homogeneous atmosphere with surface temperature $T_0 = 273$ K and surface pressure 1000 hPa. (Use the ideal gas law and hydrostatic balance.)

Solution: Now $dp/dz = -\rho g = -\rho_0 g$ for $\rho =$ Constant. Then,

$$\int_{p_0}^{0} dp = -\rho_0 g \int_0^H dz.$$

So, $p_0 = \rho_0 g H = \rho_0 R T_0$, and $H = p_0/\rho_0 g = RT_0/g = (287)(273)/(9.8) = 7.99$ km.

1.15. For the conditions of Problem 1.14, compute the variation of the temperature with respect to height. (This is referred to as an autoconvective lapse rate.)

Solution: From Problem 1.14 $p = p_0 - \rho_0 g z$, where p_0 is the surface pressure. But

$$T = \frac{p}{\rho R} = \frac{(p_0 - \rho g z)}{\rho_0 R} = \frac{p_0}{\rho_0 R} - \frac{g}{R}z = T_0 - \frac{g}{R}z \quad \text{Thus,} \;\; \Gamma = g/R \approx 34\,\mathrm{K\,km^{-1}}$$

1.16. Show that in an atmosphere with uniform *lapse rate* γ (where $\gamma \equiv -dT/dz$) the geopotential height at pressure level p_1 is given by

$$Z = \frac{T_0}{\gamma}\left[1 - \left(\frac{p_0}{p_1}\right)^{\frac{-R\gamma}{g}}\right]$$

where T_0 and p_0 are the sea level temperature and pressure, respectively.

Solution: From text Eq. (1.20) and the definition of geopotential:

$$d\ln p = -\left(\frac{g_0}{RT}\right)dZ = -\left(\frac{g_0}{R}\right)\frac{dZ}{(T_0 - \gamma Z)}$$

so that

$$\int_{p_0}^{p_1} d\ln p = -\left(\frac{g_0}{R}\right)\int_0^Z \frac{dZ}{(T_0 - \gamma Z)}.$$

Thus,

$$\ln\left(\frac{p_1}{p_0}\right) = \frac{g_0}{R\gamma}\ln\left(\frac{T_0 - \gamma Z}{T_0}\right),$$

so that

$$\left(\frac{p_1}{p_0}\right) = \left(\frac{T_0 - \gamma Z}{T_0}\right)^{\frac{g_0}{R\gamma}},$$

which may be solved for Z to give the desired result.

1.17. Calculate the 1000–500 hPa thickness for a constant lapse rate atmosphere with $\gamma = 6.5\,\mathrm{K\,km^{-1}}$ and $T_0 = 273\,\mathrm{K}$. Compare your results with the results in Problem 1.12.

Solution: From the result of Problem 1.16 (letting $p_0 = 1000\,\mathrm{hPa}$):

$$Z_T = \frac{T_0}{\gamma}\left[1 - \left(\frac{p_0}{p_1}\right)^{\frac{-R\gamma}{g_0}}\right] = \frac{273}{6.5\times 10^{-3}}\left[1 - (2)^{-287\left(6.5\times 10^{-3}\right)/9.81}\right] = 5187\,\mathrm{m}.$$

This is about 6% less than the thickness for an isothermal atmosphere of 273 K.

1.18. Derive an expression for the variation of density with respect to height in a constant lapse rate atmosphere.

Solution: Now $\rho = p/RT$, so letting ρ_1 be density at height z, and $\rho_0, p_{0,} T_0$ be values at $z = 0$, we have

$$\rho_1 = p_1 \Big/ \left[RT_0\frac{(T_0 - \gamma Z)}{T_0}\right],$$

but from Problem 1.16 this can be rewritten as

$$\rho_1 = p_0\left(\frac{T_0 - \gamma Z}{T_0}\right)^{\frac{g_0}{R\gamma}} \Big/ \left[RT_0\frac{(T_0 - \gamma Z)}{T_0}\right] = \rho_0\left(1 - \frac{\gamma z}{T_0}\right)^{\left(\frac{g_0}{R\gamma}-1\right)},$$

where we have used the fact that $\rho_0 = p_0/RT_0$.

1.19. Derive an expression for the altitude variation of the pressure change δp that occurs when an atmosphere with constant lapse rate is subjected to a height independent temperature change δT while surface pressure remains constant. At what height is the magnitude of the pressure change a maximum if the lapse rate is $6.5\,\mathrm{K\,km^{-1}}$, $T_0 = 300$, and $\delta T = 2\,\mathrm{K}$?

Solution: From Problem 1.16:

$$\left(\frac{p_1}{p_0}\right) = \left(\frac{T_0 - \gamma Z}{T_0}\right)^{\frac{g_0}{R\gamma}}$$

and thus

$$\left(\frac{p_1 + \delta p}{p_0}\right) = \left(1 - \frac{\gamma Z}{T_0 + \delta T}\right)^{\frac{g_0}{R\gamma}},$$

or

$$\left(\frac{\delta p}{p_0}\right) = \left(1 - \frac{\gamma Z}{T_0 + \delta T}\right)^{\frac{g_0}{R\gamma}} - \left(1 - \frac{\gamma Z}{T_0}\right)^{\frac{g_0}{R\gamma}}.$$

To find the height at which δp is a maximum, we evaluate $\partial\,(\delta p)/\partial Z$ in the preceding expression and set the result to zero. Letting $\varepsilon \equiv (g_0/R\gamma) - 1$ the result becomes

$$\frac{1}{T_0 + \delta T}\left(1 - \frac{\gamma Z}{T_0 + \delta T}\right)^{\varepsilon} = \frac{1}{T_0}\left(1 - \frac{\gamma Z}{T_0}\right)^{\varepsilon},$$

which can be simplified to the form

$$\left(\frac{T_0}{T_0 + \delta T}\right)^{\frac{1}{\varepsilon}}\left(1 - \frac{\gamma Z}{T_0 + \delta T}\right) = \left(1 - \frac{\gamma Z}{T_0}\right).$$

When solved for Z, this gives

$$Z_{\max} = \frac{T_0}{\gamma}\left[1 - \left(\frac{T_0}{T_0 + \delta T}\right)^{\frac{1}{\varepsilon}}\right] \Bigg/ \left[1 - \left(\frac{T_0}{T_0 + \delta T}\right)^{1+\frac{1}{\varepsilon}}\right] = 8.806\,\mathrm{km}.$$