Chapter 7

7.7 At a certain location along the ITCZ, the surface wind at 10°N is blowing from the east–northeast (ENE) from a compass angle of 60° at a speed of 8 m s^{-1} and the wind at 7°N is blowing from the south–southeast (SSE) (150°) at a speed of 5 m s^{-1}. (a) Assuming that $\partial/\partial y >> \partial/\partial x$, estimate the divergence and the vorticity averaged over the belt extending from 7°N to 10°N.

Solution:

$$\begin{aligned} \nabla \cdot \mathbf{V} &= \frac{\partial v}{\partial y} = \frac{-8\cos 60^\circ - [+5\cos(180^\circ - 150^\circ)]}{3^\circ \times 1.11 \times 10^5 \text{ m deg}^{-1}} \\ &= \frac{-4.00 - 4.33}{3.33 \times 10^5} \\ &= -2.5 \times 10^{-5} \text{ s}^{-1} \end{aligned}$$

(b) The meridional component of the wind drops off linearly with pressure from sea level (1010 hPa) to zero at the 900-hPa level. The mixing ratio of water vapor within this layer is 20 g kg^{-1}. Estimate the rainfall rate under the assumption that all the water vapor that converges into the ITCZ in the low level flow condenses and falls as rain.

Solution: The rainfall rate is given by

$$RR = -\{\nabla \cdot \mathbf{V}\} \frac{\delta p}{g} w$$

where

$$\{\nabla \cdot \mathbf{V}\} = \frac{\nabla \cdot \mathbf{V}}{2}$$

Substituting values yields

$$\begin{aligned} RR &= 2.5 \times 10^{-5} \text{ s}^{-1} \times \frac{\frac{1}{2}(1010 - 900) \times 10^2 \text{ Pa}}{9.8 \text{ m s}^{-2}} \times 20 \times 10^{-3} \\ &= 2.81 \times 10^{-4} \text{ kg s}^{-1} \text{ or } 2.42 \text{ kg day}^{-1} \end{aligned}$$

which is equivalent to a layer of liquid water 2.42 cm deep.

7.8 Consider a velocity field that can be represented as

$$\mathbf{V}_{\Psi} = -\mathbf{k} \times \nabla \Psi$$

or, in Cartesian coordinates,

$$u_\Psi = -\partial \Psi / \partial y; \; v_\Psi = \partial \Psi / \partial x$$

where Ψ is called the *streamfunction*. Prove that $\text{Div}_H \mathbf{V}$ is everywhere equal to zero and the vorticity field is given by

$$\zeta = \nabla^2 \Psi. \tag{7.43}$$

Proof: The divergence

$$\begin{aligned} \nabla \cdot \mathbf{V} &= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \\ &= \frac{\partial}{\partial x}\left(-\frac{\partial \Psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \Psi}{\partial x}\right) \\ & \quad 0 \end{aligned}$$

The vorticity

$$\begin{aligned} \zeta &= \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \\ &= \frac{\partial}{\partial x}\left(\frac{\partial \Psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(-\frac{\partial \Psi}{\partial y}\right) \\ &= \frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} \\ &= \nabla^2 \Psi \end{aligned}$$

7.9 For streamfunctions Ψ with the following functional forms, sketch the velocity field $\mathbf{V}_\Psi$. (a) $\Psi = my$, (b) $\Psi = my + n\cos 2\pi x/L$, (c) $\Psi = m(x^2 + y^2)$, and (d) $\Psi = m(xy)$ where m and n are constants.

(a) Pure zonal flow: eastward if $m < 0$ and westward if $m > 0$.

(b) A wavy zonal flow in which the wavelength of the waves is L.

(c) Solid body rotation: counterclockwise if $m > 0$.

(d) Deformation: if $m > 0$ the y axis is the axis of stretching; if $m < 0$ the x axis is the axis of stretching.

7.10 For each of the flows in the previous exercise, describe the distribution of vorticity.

(a) $\partial^2\Psi/\partial x^2 = \partial^2\Psi/\partial y^2 = 0$; $\zeta = 0$ everywhere

(b) $\partial^2\Psi/\partial x^2 = (2\pi/L)^2\, n\cos 2\pi x/L$; $\partial^2\Psi/\partial y^2 = 0$;
$\zeta = (2\pi/L)^2\, n\cos 2\pi x/L$; sinusoidal in x, independent of y.

(c) $\partial^2\Psi/\partial x^2 = 2m$; $\partial^2\Psi/\partial y^2 = 2m$; $\zeta = 4m$ everywhere

(d) $\partial^2\Psi/\partial x^2 = \partial^2\Psi/\partial y^2 = 0$; $\zeta = 0$ everywhere

7.11 Apply Eq. (7.5), which describes the advection of a passive tracer ψ by a horizontal flow pattern to a field in which the initial conditions are $\psi = -my$.

(a) Prove that at the initial time $t = 0$,

$$\frac{d}{dt}\left(\frac{\partial \psi}{\partial x}\right) = m\frac{\partial v}{\partial x} \qquad \text{and} \qquad \frac{d}{dt}\left(-\frac{\partial \psi}{\partial y}\right) = -m\frac{\partial v}{\partial y}$$

Proof: At the initial time,

$$\begin{aligned}\frac{\partial \psi}{\partial t} &= -u\frac{\partial \psi}{\partial x} - v\frac{\partial \psi}{\partial y} \\ &= 0 + mv\end{aligned}$$

and

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial \psi}{\partial x}\right) &= \frac{d}{dx}\left(\frac{\partial \psi}{\partial t}\right) \\ &= \frac{\partial}{\partial x}mv \\ &= m\frac{\partial v}{\partial x}\end{aligned} \tag{1}$$

and

$$\begin{aligned}\frac{d}{dt}\left(-\frac{\partial \psi}{\partial y}\right) &= -\frac{d}{dy}\left(\frac{\partial \psi}{\partial t}\right) \\ &= -\frac{\partial}{\partial y}mv \\ &= -m\frac{\partial v}{\partial y}\end{aligned} \tag{2}$$

Eq. (1) corresponds to the shear flow in Fig. 7.4b and Eq. (2) correponds to the deformation flow in Fig. 7.4a.

(b) Prove that for a field advected by the pure deformation flow in Fig. 7.4a, the meridional gradient $-\partial\psi/\partial y$ grows exponentially with time, whereas for a field advected by the shear flow in Fig. 7.4b, $\partial\psi/\partial x$ increases linearly with time.

Proof: From (2)

$$\frac{dm}{dt} = -m\frac{\partial v}{\partial y}$$

Hence,

$$\frac{1}{m}\frac{dm}{dt} = -\frac{\partial v}{\partial y}$$

Integrating, we obtain

$$m = m_0 e^{-\frac{\partial v}{\partial y}(t-t_0)}$$

where $m = m_0$ at time $t = t_0$.

The shear flow in Fig. 7.4b does not change m. Hence, it is evident from (1) that $\partial\psi/\partial y$ grows linearly with time.

7.12 Prove that for a flow consisting of pure rotation, the circulation C around circles concentric with the axis of rotation is equal to 2π times the angular momentum per unit mass.

Proof: From (7.3)

$$C \equiv \oint V_s ds$$

For pure rotation, $V_s = \omega r$, where r is the radius and V_s is the tangential wind speed. Hence,

$$\begin{aligned} C &= \omega r \oint ds \\ &= \omega r(2\pi r) \\ &= 2\pi \times \omega r^2 \end{aligned}$$

where ωr^2 is the angular momentum per unit mass.

7.13 Extend Fig. 7.8 by adding the positions of the marble at points 13–24.

Solution: The motion in the inertial frame of reference is periodic with period $2\pi/\Omega$. Points 13-24 correspond to the second half of the first cycle, when the marble is moving back across the dish (upward in Fig. 7.8). Hence. point 13 will coincide with the black point 11, 13 with the black point 10.... and point 24 with zero. The motion in the inertial frame of reference is periodic with period π/Ω. Points 13-24 correspond to the second circuit around the inertia circle. Hence, 13 coincides with the blue point 1, 14 with the blue point 2... and 24 with 0.

7.14 Consider two additional "experiments in a dish" conducted with the apparatus described in Fig. 7.7.

(a) The marble is released from point r_0 with initial counterclockwise motion Ωr_0 in the fixed frame of reference.

Analysis: In the inertial frame of reference, the outward centrifugal force $\Omega^2 r_0$ just balances the inward pull of gravity $g\frac{dz}{dr}$ so there is no radial motion: i.e., the motion is circular, concentric with the exis or rotation, with radius $r = r_0$. There are no tangential forces, so the tangential velocity remains constant at the initial value Ωr_0. The marble will complete one circuit around the dish at time $2\pi r_0/\Omega r_0 = 2\pi/\Omega$, which is equivalent to one period of rotation of the dish. It follows that the marble and the surface of the dish beneath it are moving at the same rate. In the rotating frame of reference the marble is stationary. Its motion may be described as an inertia circle of zero radius.

(b) The marble is released from point r_0 with initial clockwise motion Ωr_0 in the fixed frame of reference. Show that in the rotating frame of reference the marble remains stationary at the point of release.

Analysis: In the rotating frame of reference, the motion is circular, with radius r_0 as in (a) but in the clockwise direction. Hence, relative to the surface of the dish beneath it, the marble is moving clockwise with velocity $2\Omega r_0$. Based on the analysis in the text, it follows that the radius of the inertia circle in rotating coordinates is r_0. Since the motion is initially tangential and in the clockwise direction about the axis of rotation at radius r_0, the marble circles clockwise around the axis of rotation, making two complete circuits for each rotation of the turntable. For example, at $t = 6$ "hours," the marble has completed 1/4 (clockwise) circuit in the ineritial frame of reference and 1/2 (clockwise) circuit in the rotating frame of reference.

7.15 Prove that for a small closed loop of area A that lies on the surface of a rotating spherical planet, the circulation associated with the motion in an inertial frame of reference is $(f + \zeta)A$.

Proof: The circulation in an absolute frame of reference is equal to the sum of the circulation in a relative frame of reference plus the circulation associated with the rotation of the coordinate system itself; i.e.,

$$C_{abs} = C_{rel} + C_{coords} \tag{1}$$

From (7.3) it follows that if the loop is infinitesimally small, so that the relative vorticity ζ can be considered to be uniform, $C_{rel} = \zeta A$. For the special case of motion close to the poles, where the Earth's surface is normal to the axis of rotation, it follows from Exercise 7.1 that $C_{ccords} = 2\Omega A$. More generally, at latitude ϕ, the solid body rotation of the Earth's surface can be resolved into a component $2\Omega \sin\phi$ in the plane perpendicular to the Earth's rotation and a component $2\Omega\cos\phi$ in the plane of the axis of rotation. The vorticity and the circulation reside in the perpendicular component. Hence, $C_{coords} = 2\Omega \sin\phi = f$. Substituting for C_{rel} and C_{coords} in (1) yields

$$C_{abs} = (\zeta + f)\, A$$

7.16 An air parcel is moving westward at 20 m s^{-1} along the equator.

(a) Compute the apparent acceleration toward the center of the Earth from the point of view of an observer external to the Earth and in a coordinate system rotating with the Earth.

Solution: An observer external to the Earth would view the forces in an inertial coordinate system in which the centripetal acceleration is

$$a = \frac{(\Omega R_E + u)^2}{R_E} \tag{1}$$

$(\Omega R_E + u)^2 / R_E$ where u is the zonal velocity along the equator. Substituting values, we obtain

$$\begin{aligned} \frac{\left[(7.29 \times 10^{-5}) \times (6.37 \times 10^{6}) - 20\right]^2}{(6.37 \times 10^{6})} &= \frac{[4xx - 20]^2}{(6.37 \times 10^6)} \\ &= 3.11 \times 10^{-4} \text{ m s}^{-2} \end{aligned}$$

In a coordinate system rotating with the Earth, the apparent centripetal acceleration is u^2/R_E. Substituting values, we obtain

$$\frac{(20)^2}{(6.37 \times 10^6)} = 0.627 \times 10^{-4} \text{ m s}^{-2}$$

(b) Compute the apparent Coriolis force in the rotating coordinate system.

Solution: The Coriolis force is in the plane of the equator and it is directed toward the right of the motion (as viewed from a northern hemisphere perspective. Since the velocity is westward, the Coriolis force must be directed downward toward the center of the Earth. The magnitude of the Coriolis force is $2\Omega u$. substituting values we obtain

$$2 \times 7.29 \times 10^{-5} \times 20 = 29.1 \times 10^{-4} \text{ m s}^{-2}$$

Additional comment: Note that (1) may be written as

$$a = \Omega^2 R_E + 2\Omega u + u^2/R_E$$

in which the first term on the right hand side represents the contribution of the earth's rotation, the second represents the Coriolis force, and the third represents the centripetal acceleration due to the zonal motion relative to the rotating Earth. Note that in this example the second (linear) term is ~50 times as large as the third (quadratic) term.

7.17 A projectile is fired vertically upward with velocity w_0 from a point on Earth. (a) Show that in the absence of friction the projectile will land at a distance

$$\frac{4w_0^3\Omega}{3g^2}\cos\phi$$

to the west of the point from which it was fired.

Solution: The equation of motion for the vertical component of the motion w is

$$\frac{dw}{dt} = -g \tag{1}$$

The component of the vertical motion that is directed normal to the Earth's axis induces a zonally-directed Coriolis force $2\Omega w$ directed toward the right of the vertical velocity vector when viewed from a northern hemisphere perspective. Upward motion induces a westward Coriolis force and vice versa. Hence, the equation of motion for the zonal wind component is

$$\frac{du}{dt} = -2\Omega w \cos\phi \tag{2}$$

Integrating (1) yields

$$w = w_0 - gt \tag{3}$$

where w_0 is the initial vertical velocity of the projectile and g is effective gravity. Noting that $w = dz/dt$ and integrating again we obtain

$$z = w_0 t - \frac{1}{2} g t^2 \tag{4}$$

where z is height relative to the level from which the projectile was fired. Setting z equal to zero in (4) and solving for t, we obtain an expression for the length of time required for the projectile to attain its maximum height, level off, and fall back to its initial level

$$t_1 = \frac{2w_0}{g} \tag{5}$$

Now let us consider the zonal velocity and displacement of the projectile. Combining (2) and (3) yields

$$\frac{du}{dt} = -2\Omega \cos\phi \, (w_0 - gt) \tag{6}$$

and integrating yields

$$u = -2\Omega \cos\phi w_0 t + \Omega \cos\phi g t^2$$

Noting that $u = dx/dt$ and integrating (7) over the length of time that the projectile remains in the air yields

$$\begin{aligned} x &= -2\Omega \cos\phi w_0 \int_0^{t_1} t dt + \Omega \cos\phi g \int_0^{t_1} t^2 dt \\ &= -\Omega \cos\phi w_0 t_1^2 + \frac{\Omega \cos\phi g}{3} t_1^3 \end{aligned}$$

Substituting from (5) for t_1 yields

$$\begin{aligned} x &= -\frac{4\Omega \cos\phi w_0^3}{g^2} + \frac{8\Omega \cos\phi w_0^3}{3g^2} \\ &= -\frac{4\Omega w_0^3}{3g^2} \cos\phi \end{aligned} \tag{7}$$

(b) Calculate the displacement for a projectile fired upward on the equator with a velocity of 500 m s^{-2}.

Solution: Substituting values $w_0 = 500$ m s^{-1}, $\phi = 0°$, $\Omega = 7.29 \times 10^{-5}$ s^{-1}, and $g = 9.8$ m s^{-2} into (7) yields a westward displacement of 126 m.

7.18 A locomotive with a mass of 2×10^4 kg is moving along a straight track at 40 m s^{-1} at 43°N. Calculate the magnitude and direction of the transverse horizontal force on the track.

Solution: The Coriolis force is transverse to the train tracks and directed toward the right of the motion of the train in the northern hemisphere and toward the left in the southern hemisphere. Its magnitude in units of N is given by mfV, where m is the mass of the locomotive, $f = 2\Omega \sin\phi$ is the Coriolis parameter at the latitude ϕ of the locomotive, and V is the velocity of the locomotive. Substituting values $m = 2 \times 10^4$ kg, $f = 10^{-4}$ s^{-1}, and $V = 40$ m s^{-1} yields 80 N. To put this force into perspective, the ratio of this force to the weight of the train is

$$\frac{mfV}{mg} = \frac{fV}{g} \sim \frac{10^{-4} \times 40}{10} = 0.0004$$

7.19 Within a local region near 40°N, the geopotential height contours on a 500-hPa chart are oriented east–west and the spacing between adjacent contours (at 60-m intervals) is 300 km, with geopotential height decreasing toward the north. Calculate the direction and speed of the geostrophic wind.

Solution: The pressure gradient force $\mathbf{P}$ is directed northward across the isobars and its magnitude is given by $g\,(\partial Z/\partial n)$ where g is the gravitational acceleration, Z is geopotential height, and n is the direction (locally) normal to the isobars. The derivative $\partial Z/\partial n$ can be interpreted as the geometric slope of the isobars. In this example, geopotental height decreases at a rate of 60 m per 300 km or 2×10^{-4}. For geostrophic balance, $fV_g = |\mathbf{P}|$ or $V_g = |\mathbf{P}|/f$. Substituting values yields

$$V_g = \frac{2 \times 10^{-4}}{2\Omega \sin 40^\circ} = \frac{2 \times 10^{-4}}{0.95 \times 10^{-4}} = 21 \text{ m s}^{-1}$$

Since the isobars are oriented east-west, with lower pressure toward the pole, the geostrophic wind is from the west.

7.20 (a) Derive an expression for the divergence of the geostrophic wind (7.45) and give a physical interpretation of this result.

Proof: Laking use of (7.15b),

$$\begin{aligned}
\nabla \cdot \mathbf{V}_g &\equiv \frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y} \\
&= \frac{\partial}{\partial x}\left(-\frac{1}{f}\frac{\partial \Phi}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{1}{f}\frac{\partial \Phi}{\partial x}\right) \\
&= -\frac{\partial^2 \Phi}{\partial x \partial y} + \frac{\partial^2 \Phi}{\partial x \partial y} + \frac{\partial \Phi}{\partial x}\frac{\partial}{\partial y}(1/f) \\
&= \frac{\partial \Phi}{\partial x} \times -\frac{1}{(2\Omega \sin\phi)^2} \times 2\Omega \frac{\partial}{\partial y}\sin\phi \\
&= -v_g \frac{1}{(2\Omega \sin\phi)} \times 2\Omega \cos\phi \frac{\partial \phi}{\partial y}
\end{aligned}$$

Since $dy = R_E d\phi$, it follows that $\partial\phi/\partial y = 1/R_E$. Hence,

$$\nabla \cdot \mathbf{V}_g \equiv -v_g \frac{\cot\phi}{R_E} \tag{1}$$

(b) Calculate the divergence of the geostrophic wind at 45°N at a point where $v_g = 10$ m s^{-1}.

Solution: Substituting numerical values $\phi = 45°$ $v_g = 10$ m s^{-1} and $R_E = 6.37 \times 10^6$ m into (1) yields $\nabla \cdot \mathbf{V}_g = 1.57 \times 10^6$ s^{-1}.

7.21 Two moving ships passed close to a fixed weather ship within a few minutes of one another. The first ship was steaming eastward at a rate of 5 m s^{-1} and the second was steaming northward at 10 m s^{-1}. During the 3-h period that the ships were in the same vicinity, the first recorded a pressure rise of 3 hPa while the second recorded no pressure change at all. During the same 3-h period, the pressure rose 3 hPa at the location of the weather ship (50°N, 140°W). On the basis of these data, calculate the geostrophic wind speed and direction at the location of the weather ship.

Solution: Applying Eq. (1.3) to sea-level pressure p_0 yields

$$\frac{\partial p_0}{\partial t} = \frac{dp_0}{dt} - u\frac{\partial p}{\partial x} - v\frac{\partial p_0}{\partial y} \tag{1}$$

The vertical advection term does not appear because p_0 is a function of x and y only. In this book, Eq. (1.1) is usually applied to moving air parcels, but it is equally applicable to moving observing platforms, such as ships, in which case, the velocities correspond to those of the platforms. The local time rate of change of sea-level pressure can be estimated on the basis of the data at the stationary weather ship; i.e.,

$$\frac{\partial p_0}{\partial t} = \frac{+3 \text{ hPa}}{10.8 \times 10^3 \text{ s}} \tag{2}$$

Applying (1) to the eastward-moving ship yields

$$\frac{+3 \text{ hPa}}{10.8 \times 10^3 \text{ s}} = \frac{+3 \text{ hPa}}{10.8 \times 10^3 \text{ s}} - 5 \text{ m s}^{-1} \times \frac{\partial p_0}{\partial x} \tag{3}$$

from which it follows that $\partial p/\partial x = 0$ and $v_g = 0$. Applying (1) and (2) to the northward-moving ship yields

$$\frac{+3 \text{ hPa}}{10.8 \times 10^3 \text{ s}} = 0 - \left(10 \text{ m s}^{-1} \times \frac{\partial p_0}{\partial y}\right)$$

from which it follows that

$$\frac{\partial p_0}{\partial y} = \frac{-300 \text{ Pa}}{10.8 \times 10^3 \text{ s} \times 10 \text{ m s}^{-1}} = -2.78 \times 10^{-3} \text{ Pa m}^{-1}$$

From the geostrophic wind equation

$$\begin{aligned} u_g &= -\frac{1}{\rho f}\frac{\partial p_0}{\partial y} = -\frac{\partial p_0/\partial y}{\rho f} \\ &= \frac{2.78 \times 10^{-3}}{1.25 \times 2 \times 7.29 \times 10^{-5} \sin 50^\circ} \\ &= 19.9 \text{ m s}^{-1} \end{aligned}$$

7.22 At a station located at 43°N, the surface wind speed is 10 m s^{-1} and is directed across the isobars from high toward low pressure at an angle $\psi = 20°$. Calculate the magnitude of the frictional drag force and the horizontal pressure gradient force (per unit mass).

Solution: The scalar magnitude of the Coriolis force $\mathbf{C}$ is given by fV where f is the Coriolis parameter and V is the wind velocity. From Fig. 7.10 it is evident that the scalar magnitude of the frictional drag force $\mathbf{F}_s$ is $|\mathbf{C}| \cot \psi$ and the scalar magnitude of the pressure gradient force $\mathbf{P}$ is $|\mathbf{C}| \tan \psi$. At 43°N, $f = 10^{-4}$ s^{-1}. Hence, $|\mathbf{C}| = 10^{-3}$ m s^{-2}, $|\mathbf{F}_s| = 3.63 \times 10^{-4}$ m s^{-2}, and $|\mathbf{P}| = 1.06 \times 10^{-3}$ m s^{-2}.

7.23 Show that if frictional drag is neglected, the horizontal equation of motion can be written in the form

$$\frac{d\mathbf{V}}{dt} = -f\mathbf{k} \times \mathbf{V}_a \tag{7.46}$$

where $\mathbf{V}_a \equiv \mathbf{V} - \mathbf{V}_g$ is the *ageostrophic* component of the wind.

Proof: In the absence of friction, the horizontal equation of motion is

$$\frac{d\mathbf{V}}{dt} = -f\mathbf{k} \times \mathbf{V} - \nabla\Phi \tag{1}$$

From the definition of the geostrophic wind

$$\mathbf{V}_g = -\frac{1}{f}\mathbf{k} \times \nabla\Phi \tag{7.15a}$$

it follows that

$$\nabla\Phi = f\mathbf{k} \times \mathbf{V}_g \tag{2}$$

Combining (1) and (2) yields

$$\begin{aligned} \frac{d\mathbf{V}}{dt} &= -f\mathbf{k} \times \mathbf{V} - f\mathbf{k} \times \mathbf{V}_g \\ &= -f\mathbf{k} \times (\mathbf{V} - \mathbf{V}_g) \\ &= -f\mathbf{k} \times \mathbf{V}_a \end{aligned}$$

7.24 Show that in the case of anticyclonic air trajectories, gradient wind balance is possible only when

$$P_n \leq f^2 R_T/4$$

Proof: Gradient wind balance requires that

$$\frac{V^2}{R_T} + fV + |\mathbf{P}| = 0$$

From the quadratic formula,

$$V = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where $a = V^2/R_T$, $b = f$ and $c = P_n$. Real solutions exist only if $b^2 - 4ac > 0$, i.e., if

$$f^2 > 4\,|\mathbf{P}|\,/R_T$$

or

$$|\mathbf{P}| \leq f^2 R_T/4$$

7.25 Prove that the thermal wind equation can also be expressed in the forms

$$\frac{\partial \mathbf{V}_g}{\partial p} = -\frac{R}{fp}\mathbf{k} \times \nabla T$$

and

$$\frac{\partial \mathbf{V}_g}{\partial z} = \frac{g}{fT}\mathbf{k}\times\nabla T + \frac{1}{T}\frac{\partial T}{\partial z}\mathbf{V}_g$$

Proof: To obtain the first expression, we differentiate the geostrophic wind equation (7.15a) with respect to pressure, which yields

$$\frac{\partial \mathbf{V}_g}{\partial p} = \frac{1}{f}\mathbf{k} \times \nabla \frac{\partial \Phi}{\partial p} \quad (1)$$

Substituting from the hypsometric equation

$$\frac{\partial \Phi}{\partial p} = -\frac{RT}{p}$$

yields the thermal wind equation in (x, y, p) coordinates

$$\frac{\partial \mathbf{V}_g}{\partial p} = -\frac{R}{fp}\mathbf{k} \times \nabla T \quad (2)$$

To obtain the second equation we replace $\nabla\Phi$ in (7.15a) by $\alpha\nabla p$, making use of the hydrostatic equation (3.18), which yields a definition of the geostrophic wind in height coordinates

$$\mathbf{V}_g = \frac{1}{f}\mathbf{k}\times\alpha\boldsymbol{\nabla} p \quad (3)$$

Differentiating (2) with respect to height yields

$$\frac{\partial \mathbf{V}_g}{\partial z} = \frac{1}{f}\mathbf{k}\times\left(\alpha\nabla\frac{\partial p}{\partial z} + \frac{\partial \alpha}{\partial z}\nabla p\right)$$

Substituting for $\partial p/\partial z$ from the hydrostatic equation, we obtain

$$\begin{aligned}\frac{\partial \mathbf{V}_g}{\partial z} &= -\frac{g}{f\rho}\mathbf{k}\times\nabla\rho + \frac{1}{f}\mathbf{k}\times\nabla p \\ &= -\frac{g}{f\rho}\mathbf{k}\times\nabla\rho + \frac{1}{\alpha}\frac{\partial\alpha}{\partial z}\left(\frac{1}{f}\mathbf{k}\times\alpha\boldsymbol{\nabla}p\right)\end{aligned}$$

Making use of(1) and the identities $d\alpha/\alpha = dT/T$ and $d\rho/\rho = -dT/T$, as proven in Exercise 3.11, we obtain the thermal wind equation in (x, y, z) coordinates

$$\frac{\partial \mathbf{V}_g}{\partial z} = \frac{g}{fT}\mathbf{k}\times\nabla T + \frac{1}{T}\frac{\partial T}{\partial z}\mathbf{V}_g \tag{4}$$

Note that (2), the pressure coordinate version of the thermal wind equation is simpler and easier to interpret than (4) the height coordinate version.

7.26 (a) Prove that the geostrophic temperature advection in a thin layer of the atmosphere (i.e., the rate of change of temperature due to the horizontal advection of temperature) is given by $AV_{gB}V_{gT}\sin\theta$ where the subscripts B and T refer to conditions at the bottom and top of the layer, respectively, θ is the angle between the geostrophic wind at the two levels, defined as positive if the geostrophic wind veers with increasing height, and

$$A = \frac{f}{R\ln(p_B/p_T)} \tag{1}$$

Proof: For a thin layer, the geostropic temperature advection is given by

$$GTA = \overline{\mathbf{V}_g \cdot \nabla T} \simeq -\overline{\mathbf{V}}_g \cdot \nabla\overline{T} \tag{2}$$

where the braces denote vertical averages over the depth of the layer. From (7.20)

$$\mathbf{V}_{gT} - \mathbf{V}_{gB} = \left(\frac{R}{f}\ln\frac{p_B}{p_T}\right)\mathbf{k}\times\nabla\overline{T}$$

Hence

$$\nabla\overline{T} = -\frac{f}{R\ln(p_B/p_T)}\mathbf{k}\times(\mathbf{V}_{gT} - \mathbf{V}_{gB}) \tag{3}$$

Since the layer is thin,

$$\overline{\mathbf{V}}_g \cong (\mathbf{V}_{gT} + \mathbf{V}_{gB})/2 \tag{4}$$

Combining (1), (2), and (3) yields

$$\begin{aligned}GTA &= -\mathbf{k}\times(\mathbf{V}_{gT} - \mathbf{V}_{gB}) \qquad (5)\\ &= -A\frac{(\mathbf{V}_{gT} + \mathbf{V}_{gB})}{2}(\mathbf{V}_{gT} - \mathbf{V}_{gB})\sin\gamma\end{aligned}$$

where A is as defined in (1) and γ is the angle between the layer-mean wind vector $\overline{\mathbf{V}}_g$ and the thermal wind vector $\mathbf{V}_{gT} - \mathbf{V}_{gB}$, as depicted in the accompanying figure. Note that GTA in (4) is directly proportional to the area of the parallelogram ABCD in the accompanying figure and that this area, in turn is equivalent to $V_B V_T \sin\theta$. Hence,

$$GTA = AV_{gB}V_{gT}\sin\theta \tag{5}$$

It is also evident from the figure that the approximation (3) is valid provided that the thermal wind is fairly uniform within the layer. The layer need not necessarily be thin.

(b) At a certain station located at 43°N, the geostrophic wind at the 1000-hPa level is blowing from the southwest (230°) at 15 m s^{-1} while at the 850-hPa level it is blowing from the west–northwest (300°) at 30 m s^{-1}. Estimate the geostrophic temperature advection.

Solution: Substituting values into (1) we obtain

$$\begin{aligned} A &= \frac{10^{-4}\ \mathrm{s}^{-1}}{287\ \mathrm{J\ kg^{-1}K^{-1}}\ln(1000/850)} \\ &= 2.13\times10^{-6}\ \mathrm{m^{-2}\ s^{-1}\ K} \end{aligned}$$

and, from (5)

$$\begin{aligned} GTA &= \left(2.13\times10^{-6}\right)\ \mathrm{m^{-2}\ s^{-1}\ K}\times15\ \mathrm{m\ s^{-1}}\times30\ \mathrm{m\ s^{-1}}\times\sin70^\circ \\ &= 9.01\times10^{-4}\ \mathrm{K\ s^{-1}}\ \text{or}\ 77.8\ \mathrm{K\ day^{-1}} \end{aligned}$$[1]

The geostrophic temperature advection is positive because the wind is veering with height.

7.27 At a certain station, the 1000-hPa geostrophic wind is blowing from the northeast (050°) at 10 m s^{-1} while the 700-hPa geostrophic wind is blowing from the west (270°) at 30 m s^{-1}. Subsidence is producing adiabatic warming at a rate of 3°C day^{-1} in the 1000 to 700-hPa layer and diabatic heating is negligible. Calculate the time rate of change of the thickness of the 1000 to 700-hPa layer. The station is located at 43°N.

Answer: Decreasing at a rate of 140 m day^{-1}

7.28 Prove that in the line integral around any closed loop that lies on a pressure surface, the pressure gradient force $-\nabla\Phi$ vanishes, i.e.,

$$\oint \mathbf{P}ds = -\oint \nabla\Phi ds = 0 \tag{7.47}$$

[1] Such large time rates of temperature are observed only in very strong (and narrow) frontal zones. They are not sustained over intervals of more than a few hours.

Proof: For any line integral

$$-\int_1^2 \nabla\Phi ds = -\int_1^2 \frac{\partial\Phi}{\partial s} ds = \Phi_2 - \Phi_1$$

For a full circuit around any loop, the line integral must vanish because $\Phi_2 = \Phi_1$.

7.29 Prove that in the absence of friction, the circulation

$$C_a \equiv \oint \mathbf{c} \cdot ds \tag{7.48}$$

is conserved, where **c** is the velocity in an inertial frame of reference.

Proof: For each element ds along the loop, the equation of motion is applicable. In the absence of forces,

$$\frac{d\mathbf{c}}{dt} = 0$$

Hence, in the absence of forces, $C_a \equiv \oint \mathbf{c} \cdot ds$ is conserved. In Exercise 7.28 it was shown that the existence of a pressure gradient force does not change the circulation. It follows that in an inertial frame of reference, in which there are no apparent forces, circulation is conserved in the absence of friction.

7.30 Based on the result of the previous exercise, prove that in an inertial frame of reference

$$[c_s]\frac{dL}{dt} = -L\frac{d[c_s]}{dt}$$

where

$$[c_s] \equiv \frac{\oint \mathbf{c} \cdot ds}{L}$$

$[c_s]$ is the tangential velocity averaged over the length of the loop and $L = \oint ds$ is the length of the loop. Hence, a lengthening of the loop must be accompanied by a proportionate decrease in the mean tangential velocity along the loop and vice versa.

Proof: Based on the definition of c_s, we can write

$$C_a \equiv \oint \mathbf{c} \cdot ds = [c_s]L$$

If circulation is conserved,

$$\frac{d}{dt}[c_s]L = [c_s]\frac{dL}{dt} + L\frac{d[c_s]}{dt} = 0$$

It follows that

$$[c_s]\frac{dL}{dt} = -L\frac{d[c_s]}{dt}$$

7.31 For the special case of an axisymmetric flow in which the growing or shrinking loop is concentric with the axis of rotation, prove that the conservation of circulation is equivalent to the conservation of angular momentum. [**Hint:** Show that angular momentum $M = C/2\pi$.]

See the solution to Exercise 7.12. The exercise was inadvertently repeated.

7.32 Solution in text

7.33 Consider a sinusoidal wave along latitude ϕ with wavelength L and amplitude v in the meridional wind component. The wave is embedded in a uniform westerly flow with speed U. (a) Show that the amplitude of the geopotential height perturbations associated with the wave is $fvL/2\pi g$ where f is the Coriolis parameter and g is the gravitational acceleration. (b) Show that the amplitude of the associated vorticity perturbations is $(2\pi/L)v$. Show that the maximum values of the advection of planetary and relative vorticity are βv and $(2\pi/L)^2 Uv$, respectively, and that they are coincident and of opposing sign. (c) Show that the advection terms exactly cancel for waves with wavelength

$$L_S = 2\pi\sqrt{\frac{U}{\beta}}.$$

(a) If the meridional wind component is in geostrophic balance,

$$v = v_m \cos(2\pi x/L) = \frac{g}{f}\frac{\partial Z}{\partial x}$$

so

$$\begin{aligned} Z &= \frac{f}{g}\int v\cos(2\pi x/L)\,dx \\ &= \frac{f v_m L}{2\pi g}\sin(2\pi x/L) \end{aligned}$$

and

$$Z_m = \frac{f v_m L}{2\pi g} \qquad (1)$$

The vorticity is given by

$$\begin{aligned} \zeta &= \frac{\partial}{\partial x} v_m \cos(2\pi x/L) \qquad (2) \\ &= -\frac{2\pi}{L} v_m \sin(2\pi x/L) \end{aligned}$$

The rate of advection of relative vorticity by the zonal wind is

$$\begin{aligned} -U\frac{\partial \zeta}{\partial x} &= -U\frac{\partial}{\partial x}\left[-\frac{2\pi}{L} v_m \sin(2\pi x/L)\right] \qquad (3) \\ &= \frac{(2\pi)^2 U v_m}{L^2}\cos(2\pi x/L) \end{aligned}$$

The rate of advection of planetary vorticity is given by

$$-\beta v_m \cos(2\pi x/L) \tag{4}$$

The rates of advection of relative and planetary vorticity are thus of the same functional form: for waves of any given zonal wavelength L, they are linear multiples of one another. For the special case of stationary waves with zonal wavelength $L = L_S$, the advection of absolute (i.e., relative plus planetary) vorticity is zero, so the rates prescribed by (3) and (5) are equal and opposite; i.e.,

$$(2\pi/L_S)^2 U v_m = \beta v_m$$

$$L_S = 2\pi\sqrt{\frac{U}{\beta}}.$$

L_S is referred to as the wavelength of a stationary Rossby wave. For $L < L_S$, the advection of relative vorticity is larger than the advection of planetary vorticity and the waves propagate eastward and for $L > L_S$, the advection of planetary vorticity dominate and the waves propagate westward. This behavior is actually observed when the waves propagating along a prescribed latitude circle such as 50°N are decomposed into wave-numbers by harmonic analysis. Zonal wavenumber 1 tends to propagate westward and wavenumbers 4 and higher propagate eastward.

7.34 Suppose that the wave in the previous exercise propagating is along 45° latitude and has a wavelength of 4000 km. The amplitude of the meridional wind perturbations associated with the wave is 10 m s^{-1} and the background flow $U = 20$ m s^{-1}. Assume that the velocity field is independent of latitude and neglect the latitude dependence of the Coriolis parameter f. Using the results of the previous exercise, estimate (a) the amplitude of the geopotential height and vorticity perturbations in the waves, (b) the amplitude of the advection of planetary and relative vorticity, and (c) the wavelength of a stationary Rossby wave embedded in a westerly flow with a speed of 20 m s^{-1} at 45° latitude.

Answers: (a) 64.6 m; 1.57×10^{-5} s^{-1} (b) 1.62×10^{-10} and 4.93×10^{-10} s^{-2}, respectively; and (c) 7000 km

7.35 Verify the validity of the conservation of barotropic potential vorticity $(f + \zeta)/H$, as expressed in Eqn. (7.27).

From Eq. (7.21b)

$$\frac{d}{dt}(f + \zeta) = -(f + \zeta)\nabla \cdot \mathbf{V}$$

From (7.2) and (7.26)

$$\frac{1}{H}\frac{dH}{dt} = -\nabla \cdot \mathbf{V}$$

where H is the depth of a column of fluid. Combining these expressions yields

$$\frac{1}{(f+\zeta)}\frac{d}{dt}(f+\zeta) - \frac{1}{H}\frac{dH}{dt} = 0$$

Making use of the identity $dy/y = -dx/x$, where $y = 1/x$, the above expression can be rewritten as

$$\frac{1}{(f+\zeta)}\frac{d}{dt}(f+\zeta) + \frac{1}{(1/H)}\frac{d(1/H)}{dt} = 0$$

from which it follows that

$$\frac{d}{dt}\left(\frac{f+\zeta}{H}\right) = 0$$

7.36 Consider barotropic ocean eddies propagating meridionally along a sloping continental shelf, with depth increasing toward the east, as pictured in Fig. 7.25, conserving barotropic potential vorticity in accordance with (7.27). There is no background flow. In which direction will the eddies propagate?

At point A, the flow in the eddies is toward shallower water so the depth H of a moving column of water is decreasing. Absolute vorticity $(f+\zeta)$ decrease by a proportionate amount so that barotropic potential vorticity $(f+\zeta)/H$ remains constant. The planetary vorticity does not change appreciably because the water column at A is not moving meridionally. Hence there must be an anticyclinic tendency in relative vorticity ζ. By the same line of reasoning it can be argued that there must be a cyclonic vorticity tendency at pont B, where water columns are being stretched as they move away from the shore.

The direction of propagation of the eddies depends on which hemisphere they are in. If they are in the northern hemisphere, the clockwise eddy situated between A and B is anticyclonic. This eddy must be moving toward point A, where the vorticity tendency is negative, and away from point B where the vorticity tendency is positive. Hence the eddies must be propagating southward. By the same reasoning, if the eddies are in the southern hemisphere they must be propagating northward. Note that in both cases the direction of propagation is equatorward.

Analogous behavior is observed in atmospheric waves propagating along sloping terrain, as discussed in Section 8.2.

Barotropic ocean eddies propagating along a continental shelf, as envisioned in Exercise 7.36.

7.37 During winter in middle latitudes, the meridional temperature gradient is typically on the order of 1° per degree of latitude, while potential temperature increases with height at a rate of roughly 5°C km^{-1}. What is a typical slope of the potential temperature surfaces in the meridional plane? Compare this result with the slope of the 500-hPa surface in Exercise 7.19.

In the meridional plane

$$d\theta = \frac{\partial\theta}{\partial y}dy + \frac{\partial\theta}{\partial z}dz$$

On a potential temperature surface $d\theta = 0$. Hence,

$$\frac{dz}{dy} = -\frac{\partial\theta/\partial y}{\partial\theta/\partial y}$$

Substituting values yields

$$\frac{dz}{dy} = \frac{1 \text{ K per } 111 \text{ km}}{5 \text{ K per km}} \sim \frac{0.01}{5} = 2 \times 10^{-3}$$

Comparing with Exercise 7.19, we find that the isentropic in extratropical latitudes surfaces slope about an order of magnitude more steeply than the pressure surfaces

7.38 Making use of the approximate relationship $\omega \simeq -\rho g w$ show that the vertical motion term

$$(\frac{\kappa T}{p} - \frac{\partial T}{\partial p})\omega$$

in Eqn. (7.37) is approximately equal to

$$-w(\Gamma_d - \Gamma)$$

where Γ_d is the dry adiabatic lapse rate and Γ the observed lapse rate.

The vertical motion term is approximately equal to

$$-(\frac{\kappa T}{p} - \frac{\partial T}{\partial p})\rho g w$$

From the equation of state

$$\frac{\kappa T}{p} = \frac{RT}{c_p p} = \frac{\alpha}{c_p}$$

From the hydrostatic equation

$$-\frac{\partial T}{\partial p} = \frac{1}{\rho g}\frac{\partial T}{\partial z}$$

Substituting yields

$$-\left(\frac{\alpha}{c_p} + \frac{1}{\rho g}\frac{\partial T}{\partial z}\right)\rho g w$$

Simplifying and using the definitions $\Gamma \equiv -\partial T/\partial z$ and $\Gamma_d \equiv g/c_p$ yields

$$-w(\Gamma_d - \Gamma)$$

7.40 In middle-latitude winter storms, rainfall (or melted snowfall) rates on the order of 20 mm day^{-1} are not uncommon. Most of the convergence into these storms takes place within the lowest 1–2 km of the atmosphere (say, below 850 hPa) where the mixing ratios are on the order of 5 g kg^{-1}. Estimate the magnitude of the convergence into such storms.

$$\text{Rainfall rate} = -\left(\nabla \cdot \mathbf{V}\right)\frac{\delta p}{g} w$$

where δp is the depth of the layer (expressed in units of Pa) in which the precipitation is occurring and w is the water vapor mixing ratio (expressed in dimensionless units). A rainfall rate of 20 mm day^{-1} is equivalent to a mass flux per unit area of 20 kg per 86,400 s^{-1} or 2.31×10^{-4} kg s^{-1}. Substituting values into (1) and solving for the convergence $-\left(\nabla \cdot \mathbf{V}\right)$ yields

$$\begin{aligned} -\left(\nabla \cdot \mathbf{V}\right) &= \frac{2.31 \times 10^{-4}}{\frac{(1.5\times 10^4)}{9.8} \times 5 \times 10^{-3}} \\ &= 3.02 \times 10^{-5}\ \text{s}^{-1} \end{aligned}$$

7.43 Averaged over the mass of the atmosphere, the root mean squared velocity of fluid motions is ∼17 m s^{-1}. By how much would the center of gravity of the atmosphere have to drop in order to generate the equivalent amount of kinetic energy?

The kinetic energy per unit area is given by

$$KE = \left(\frac{p_0}{g}\right)\frac{V^2}{2}$$

where V is the root mean squared velocity, p is the mean surface pressure, and g is gravity. The potential energy (per unit area) released by lowering the center of mass of the atmosphere by the increment δz is given by

$$\delta PE = \left(\frac{p_0}{g}\right)\delta\Phi = \left(\frac{p_0}{g}\right) g\delta z$$

where Φ is geopotential. Hence,

$$\delta z = \frac{V^2}{2g}$$

Substituting values, we obtain

$$\delta z = \frac{(17)^2}{2 \times 9.8} = 14.7 \text{ m}$$

7.44 Suppose that a parcel of air initially at rest on the equator is carried poleward to 30° latitude in the upper branch of the Hadley cell, conserving angular momentum as it moves. In what direction and at what speed will it be moving when it reaches 30°?

The angular momentum per unit mass of air moving with zonal velocity u is

$$M = (\Omega R_E \cos\phi + u)\, R_E \cos\phi$$

where Ω and R_E are the rotation rate and radius of the Earth. For an air parcel at rest on the equator

$$M = \Omega R_E^2$$

If the air parcel moves poleward to latitude $\phi, \Omega R_E^2$ while conserving angular momentum,

$$(\Omega R_E \cos\phi + u)\, R_E \cos\phi = \Omega R_E^2$$

Solving for the zonal velocity

$$u = \frac{\Omega R_E^2 \left(1 - \cos^2\phi\right)}{R_E \cos\phi}$$

Substituting values $\Omega = 7.29 \times 10^{-5}$ s^{-1}, $R_E = 6.37 \times 10^6$ and $\cos\phi = \sqrt{3}/2$, we obtain $u = 120$ m s^{-1}.

7.45 On average over the globe, kinetic energy is being generated by the cross-isobar flow at a rate of ∼2 W m^{-2}. At this rate, how long would it take to "spin up" the general circulation, starting from a state of rest?

The "spin up" time is

$$T = \frac{KE}{G} \tag{1}$$

where

$$KE = \left(\frac{p_0}{g}\right)\frac{V^2}{2}$$

is the kinetic energy of the atmospheric circulation and G is the rate of kinetic energy production. Assume a root-mean-squared velocity of fluid motions of 17 m s^{-1}, as in Exercise 7.43. Substituting values into (1), we obtain

$$T \sim \frac{1.5 \times 10^6 \text{ J m}^{-2}}{2 \text{ W m}^{-2}} \sim 7 \times 10^5 \text{ or about a week}$$

7.46 The following laboratory experiment provides a laboratory analog for the kinetic energy cycle in the atmospheric general circulation.

A tank with vertical walls is filled with a homogeneous (constant density) incompressible fluid. The height of the undulating free surface of the fluid is $Z(x, y)$.

(a) Since the density of the fluid is horizontally uniform, pressure decreases with height at a uniform rate as well. Hence,

$$\frac{\partial \mathbf{P}}{\partial z} = \frac{\partial}{\partial z}\left(\frac{1}{\rho}\nabla p\right) = 0$$

everywhere within the fluid. If follows that the pressure gradient force on any horizontal surface within the fluid must be exactly the same as that at the level corresponding to the low point on the free surface, where

$$\mathbf{P} = -g\nabla Z \tag{1}$$

(b) Since the density of the fluid is constant,

$$\frac{d}{dt}\delta x\delta y\delta z = 0$$

following an imaginary block of fluid as it moves about in the tank. Proceeding as in the proof of (7.2) in the text, we obtain

$$\nabla \cdot \mathbf{V} + \frac{\partial w}{\partial z} = 0 \tag{2}$$

Note that this relationship does not hold for a compressible fluid , but if the motion is hydrostatic, an analogous relation [i.e., Eq. (7.39a)] can be derived in pressure coordinates.

(c) Integrating (1) from the bottom (B) to the top (T) of the layer of fluid in the tank yields

$$w_T - w_B = -\int_B^T (\nabla \cdot \mathbf{V})\, dz$$

If we assume that $\mathbf{V}$ and $(\nabla \cdot \mathbf{V})$ are independent of height and that $w_B = 0$, the integration yields

$$w_T = \frac{dZ}{dt} = -Z\,(\nabla \cdot \mathbf{V}) \tag{3}$$

where Z is the mean height of the free surface of the fluid. From here onward we will assume that the vartations in the height of the free surface Z are small in comparison to the mean height $[Z]$ of the free surface, in which case,

$$\frac{dZ}{dt} = -[Z]\,(\nabla \cdot \mathbf{V}) \tag{4}$$

Hence, the free surface of the fluid in the tank is rising in areas of convergence and sinking in areas of divergence.

(d) The potential energy of the fluid in the tank is

$$\begin{aligned} PE &= \int\int\int \rho g z dx dy dz \\ &= \rho g \int\int\int z dx dy dz \end{aligned}$$

Integrating ftom the bottom of the tank up to the height of the free surface yields

$$\begin{aligned} PE &= \frac{1}{2}\rho g \int\int Z^2 dA \\ &= \frac{1}{2}\rho g \int\int [Z]^2 dA + \frac{1}{2}\rho g \int\int Z'^2 dA \end{aligned}$$

where $dA = dxdy$ and Z' is the difference between the local height of the free surface and the area-averaged height $[Z]$ of the fluid in the tank. Provided that the volume of fluid in the tank is constant, the first term cannot change with time. Hence, the vartable part of the potential energy, which is "available" for conversion to kinetic energy, is given by

$$APE = \frac{1}{2}\rho g \int\int Z'^2 dA \tag{5}$$

Note that available potential energy (APE) is a positive definite quantity; i.e., it is zero if the surface of the fluid in the tank is perfectly flat and it is positive if there are any variations in the height of the surface. If the fluid tank started out with a wavy free surface and if, over the course of time, the surface became flatter, PE would decrease (i.e., be released for conversion to kinetic energy). APE, as defined in (5) is the maximum possible amount of PE that could be released in this manner.

(e) The rate of release of APE may be estimated by differentiating (5) with respect to time

$$\begin{aligned} -\frac{d}{dt}APE &= -\rho g \int\int Z' \frac{dZ'}{dt} dA \qquad (6) \\ &= -\rho g \int\int Z' w'_T dA \end{aligned}$$

Hence, available potential energy will be released if the fluid is sinking in the regions of the tank where the free surface is raised relative to the reference level $[Z]$ and rising in regions of the tank in which the free surface is depressed. Vertical motions in this sense will tend to flatten the free surface, thereby reducing the APE.

(f) Combining (4) and (6) yields

$$-\frac{d}{dt}APE = -\rho g\,[Z] \iint Z'\,(\boldsymbol{\nabla}\cdot\mathbf{V})\,dA \tag{7}$$

Hence, potential energy is released if the fluid in the tank is tending to diverge (horizontally) out of areas in which the surface is raised, relative to the reference level, and converge into areas in which the surface is depressed. This is analogous to air diverging out of highs and converging into lows in the Earth's atmosphere.

(g) Since the tank is cylindrical and fluid cannot flow through the walls, it can be inferred from Gauss's theorem [i.e., from Eq. (7.4) with V_n replaced by $Z'V_n$ and $(\nabla\cdot\mathbf{V})$ by $(\nabla\cdot Z'\mathbf{V})$] that

$$\iint (\nabla\cdot\mathbf{V}Z')\,dA = 0 \tag{8}$$

Integrating by parts yields

$$\iint Z'\,(\nabla\cdot\mathbf{V})\,dA = -\iint (\mathbf{V}\cdot\boldsymbol{\nabla}Z')\,dA$$

Substituting into (7) yields

$$-\frac{d}{dt}APE = -\rho g\,[Z] \iint (\mathbf{V}\cdot\boldsymbol{\nabla}Z)dA \tag{9}$$

From (1) the pressure gradient force $\mathbf{P}$ in the horizontal equation of motion is proprtional to ∇Z and directed down the Z (or pressure) gradient from higher toward lower values. Hence, (9) indicates that potential energy is being released if, on average over the area of the tank, the horizontal velocity $\mathbf{V}$ is directed down the pressure gradient, in the direction of the pressure gradient force $\mathbf{P}$, in which case, work is being done on the fluid by the pressure gradient force.

It is possible to obtain (9) by considering the production of kinetic energy by the horizontal pressure gradient force. Start with the horizontal equation of motion [adapted from (7.14)]

$$\frac{d\mathbf{V}}{dt} = -g\nabla Z - f\mathbf{k}\times\mathbf{V} + \mathbf{F}$$

Taking the dot product with the horizontal velocity vector $\mathbf{V}$, we obtain

$$\frac{d}{dt}\frac{V^2}{2} = -g\,(\mathbf{V}\cdot\nabla Z) + \mathbf{F}\cdot\mathbf{V} \tag{10}$$

The Coriolis force does not contribute to changing the kinetic energy because it is perpendicular to the velocity vector: it contributes to changing the wind direction but not the wind speed. Integrating (10) over the mass of the fluid in the tank yields

$$\begin{aligned}\frac{d}{dt}KE &= -\rho g\,[Z]\iint(\mathbf{V}\cdot\boldsymbol{\nabla}Z)dA + -\rho g\,[Z]\iint(\mathbf{F}\cdot\mathbf{V})dA \qquad (11)\\ &= \frac{d}{dt}APE - D\end{aligned}$$

where D is the frictional dissipation of kinetic energy. In the absence of frictional dissipation

$$\frac{d}{dt}(APE + KE) = 0$$