# Chapter 4

**4.12** Remote sensing in the microwave part of the spectrum relies on radiation emitted by oxygen molecules at frequencies near 55 gHz. Calculate the wavelength and wavenumber of this radiation.

From (4.2)

$$\lambda = \frac{c^*}{\widetilde{\nu}} = \frac{3 \times 10^8 \text{ m s}^{-1}}{55 \times 10^9 \text{ s}^{-1}} = 5450\ \mu\text{m}$$

**4.13** The spectrum of monochromatic intensity can be defined either in terms of wavelength $\lambda$ or wavenumber $\nu$ such that the area under the spectrum, plotted as a linear function of $\lambda$ or $\nu$ is proportional to intensity. Show that $I_\nu = \lambda^2 I_\lambda$.

$$dI = I_\lambda d\lambda = I_\nu d\nu$$

From (4.1)

$$\nu = \frac{1}{\lambda}$$

from which it follows that

$$d\nu = -\frac{d\lambda}{\lambda^2}$$

Substituting for $d\lambda$ in the first expression, cancelling the common factor $d\nu$, and ignoring the minus sign, which is taken into account by reversing the direction of the integration, we obtain

$$I_\nu = \lambda^2 I_\lambda$$

**4.14** A body is emitting radiation with the following idealized spectrum of monochromatic flux density

$$\begin{array}{cc}
\lambda < 0.35\ \mu\text{m} & F_\lambda = 0 \\
0.35\ \mu\text{m} < \lambda < 0.50\ \mu\text{m} & F_\lambda = 1.0 \text{ W m}^{-2}\mu\text{m}^{-1} \\
0.50\ \mu\text{m} < \lambda < 0.70\ \mu\text{m} & F_\lambda = 0.5 \text{ W m}^{-2}\mu\text{m}^{-1} \\
0.70\ \mu\text{m} < \lambda < 1.00\ \mu\text{m} & F_\lambda = 0.2 \text{ W m}^{-2}\mu\text{m}^{-1} \\
\lambda > 1.00\ \mu\text{m} & F_\lambda = 0
\end{array}$$

Calculate the flux density of the radiation.

$$\begin{aligned}
F &= \int F_\lambda d\lambda = \sum_{i-1}^{N} F_{\lambda i} \triangle \lambda_i \\
&= 1.0 \times 0.15 + 0.5 \times 0.20 + 0.2 \times 0.3 \\
&= 0.15 + 0.10 + 0.06 = 0.31 \text{ W m}^{-2}
\end{aligned}$$

**4.15** An opaque surface with the following absorption spectrum is subjected to the radiation described in the previous exercise.

$$\begin{array}{ll} \lambda < 0.70\ \mu\text{m} & A_\lambda = 0 \\ \lambda > 0.70\ \mu\text{m} & A_\lambda = 1 \end{array}$$

How much of the radiation is absorbed? How much is reflected?

$$\begin{aligned} F(\text{absorbed}) &= \int A_\lambda F_\lambda d\lambda = \sum_{i-1}^{N} A_{\lambda i} F_{\lambda i} \triangle\lambda_i \\ &= 1.0 \times 1.0 \times 0.15 + 1.0 \times 0.5 \times 0.2 \\ &= 0.15 + 0.10 = 0.25\ \text{W m}^{-2} \end{aligned}$$

$$\begin{aligned} F(\text{reflected}) &= \int (1 - A_\lambda) F_\lambda d\lambda = \sum_{i-1}^{N} (1 - A_{\lambda i}) F_{\lambda i} \triangle\lambda_i \\ &= 1.0 \times 0.3 \times 0.2 \\ &= 0.06\ \text{W m}^{-2} \end{aligned}$$

**4.16** Calculate the ratios of the incident solar radiation at noon on north and south facing 5° slopes (relative to the horizon) in seasons in which the solar zenith angle is (a) 30° and (b) 60°.

For the 30° solar zenith angle the ratio

$$r = \frac{F_{\text{north facing slope}}}{F_{\text{south facing slope}}}$$

is

$$r = \frac{I \cos 35^\circ}{I \cos 25^\circ} = 0.84$$

and for the 60° zenith angle the ratio is

$$r = \frac{I \cos 65^\circ}{I \cos 55^\circ} = 0.74$$

**4.17** Compute the daily insolation at the North Pole at the time of the summer solstice when the earth-sun distance is 1.52 $\times 10^8$ km. The tilt of the earth's axis is 23.5°. Compare this value with the minimum value that occurs in association with the Earth's orbital cycles described in §2.5.3. *Answer*: 46.4 MJ m$^{-2}$ day$^{-1}$ versus xxx MJ m$^{-2}$ day$^{-1}$.

At the time of the summer solstice the incident solar radiation is independent of time of day and the solar zenoth angle is $(90^\circ - 23.5^\circ) = 66.5^\circ$. Hence the flux density is equal to

$$1368\ \text{W m}^{-2} \times \left( \frac{1.50 \times 10^8\ \text{km}}{1.52 \times 10^8\ \text{km}} \right)^2 \times \cos 66.5^\circ =$$

**4.18** Compute the daily insolation at the top of the atmosphere at the equator at the time of the equinox (a) by integrating the flux density over a 24 hour period and (b) by simple geometric considerations. Compare your result with the value in the previous exercise and with Fig. 10.7. *Answer:* 38.0 MJ $m^{-2}$ day.

**4.19** What fraction of the flux of energy emitted by the sun does the earth intercept? *Answer* 1.4 $\times 10^{-10}$.

**4.20** Show that for small variations in the earth's radiation balance

$$\frac{\delta T_E}{T_E} = \frac{1}{4}\frac{\delta F_E}{F_E}$$

where $T_E$ is the planet's equivalent blackbody temperature and $F_E$ is the flux of radiation emitted from the top of its atmosphere. Use this relationship to estimate the change in effective temperature that would occur in response to (a) the seasonal variations in the sun-earth distance due to the eccentricity of the earth's orbit (presently $\sim 3\%$),(b) an increase in the earth's albedo from 0.30 to 0.31.

From (4.12)

$$F = \sigma T^4$$

Taking the log yields

$$\ln F = 4 \ln T$$

Taking the differential yields

$$\frac{\delta F}{F} = 4\frac{\delta T}{T}$$

and dividing both sides by 4 yields

$$\frac{\delta T_E}{T_E} = \frac{1}{4}\frac{\delta F_E}{F_E} \tag{1}$$

(a) From the inverse square law

$$F_E = const \times d^{-2}$$

where $d$ is the earth-sun distance. Taking the log yields

$$\ln F_E = \ln const - 2 \ln d$$

Taking the differential yields

$$\frac{\delta F_E}{F_E} = -2\frac{\delta d}{d} \tag{2}$$

Combining (1) and (2) yields

$$\frac{\delta T_E}{T_E} = -\frac{1}{2}\frac{\delta d}{d} \tag{1}$$

Substituting $\delta d/d = 0.03$ yields $\delta T_E/T_E = -0.06$. If $T_E = 255$ K, then $\delta T_E = 1.5$ K.

(b)

$$F_E = const \times (1 - A)$$

where $A$ is the eplanetary albedo. Taking the log yields

$$\ln F_E = \ln const - \ln(1 - A)$$

Taking the differential yields

$$\frac{\delta F_E}{F_E} = \frac{\delta(1-A)}{(1-A)} \tag{2}$$

Combining (1) and (2) yields

$$\frac{\delta T_E}{T_E} = \frac{1}{4}\frac{\delta(1-A)}{(1-A)} \tag{3}$$

Substituting $\delta d/d = 0.01/0.70$ yields $\delta T_E/T_E = -0.00357$. If $T_E =$ 255 K, then $\delta T_E = 0.91$ K.

**4.21** Show that the emission the flux density of incident solar radiation on any planet in our solar system is 1366 W m$^{-2}\times$ $d^{-2}$, where $d$ is the Planet-Sun distance, expressed in astronomical units (A.U., multiples of the Earth-Sun distance).

**Method 1:** The flux density is equal to the intensity of solar radiation $I_s$ (the same for all planets) times the arc of solid angle $\delta\Omega$ subtended by the sun, as viewed from the planet, i.e.,

$$F = I_s \delta\Omega$$

where

$$\delta\Omega = 4\pi \times \left(\frac{R_s}{d}\right)^2$$

where $R_s$ is the radius of the sun and $d$ is the distance between the planet and the sun. Hence,

$$F = (4\pi I_s R_s^2) d^{-2}$$

where the factor in parentheses is the same for all planets. For Earth,

$$1368 = (....) d_E^{-2} \tag{1}$$

and for any other planet

$$F_p = (....) d_p^{-2} \tag{2}$$

Dividing (2) by (1) yields

$$\frac{F_p}{1368} = \left(\frac{d_p}{d_E}\right)^{-2} \tag{3}$$

**Method 2:** The final result above is an expression of the inverse square law which follows directly from the fact that the flux of solar radiation through all spheres concentric with the sun must be the same and hence the product of the flux density of solar radiation times distance from the sun must be the same for all planets.

**4.28** If the Moon subtends the same arc of solid angle in the sky that the sun does, and it is directly overhead, prove that the flux density of moonlight on a horizontal surface on Earth is given by $a(R_s/d)^2$ where $a$ is the Moon's albedo, $R_s$ the radius of the Sun, and $d$ is the Earth-Sun distance. Estimate the flux density of moonlight under these conditions, assuming a lunar albedo of 0.07.

*Answer*

**4.29** Suppose that the Sun's emission or the earth's albedo were to change abruptly by a small increment. Show that the *radiative relaxation rate* for the atmosphere (i. e, the initial rate at which the earth's equivalent blackbody temperature would respond to the change, assuming that the atmosphere is thermally isolated from the other components of the Earth system), is given by

$$\frac{dT}{dt} = -\frac{4\sigma T^3 \delta T_E}{c_p p_s g^{-1}}$$

where $\delta T$ is the initial departure of the equivalent blackbody temperature from radiative equilibrium), $\sigma$ is the Stefan-Boltzmann constant, $T_E$ is the equivalent blackbody temperature in K, $c_p$ is the specific heat of air, $p_s$ is the global-mean surface pressure and $g$ is the gravitational acceleration. The time $\delta T(dT/dt)^{-1}$ required for the atmosphere to fully adjust to the change in radiative forcing, if this initial time rate of change of temperature were maintained until the new equilibrium was established is called the *radiative relaxation time.* Estimate the radiative relaxation time for the earth's atmosphere.

*Answer*

**4.30** A small, perfectly black, spherical satellite is in orbit around the Earth at an altitude of 2000 km as depicted in Fig. 4.37. What angle does the earth subtend when viewed from the satellite?

The integration is most easily performed using a spherical coordinate system centered on the satellite with the zenith pointed toward the center of the Earth. In this coordinate system, the arc of solid angle corresponds to

$$\begin{aligned}
\triangle\omega &= 2\pi \int_0^{\psi} \sin\theta d\theta \\
&= 2\pi(1 - \cos\psi)
\end{aligned}$$

where $\psi$ is the angle between the zenith and the limb of the Earth (ie., a straight line passing through the satellite and tangent to the circle that defines the Earth). It is readily verified that

$$\psi = \sin^{-1}\frac{R_E}{d}$$

where $R_E$ is the radius of the Earth and $d$ is the distance between the center of the satellite and the center of the Earth. Substituting values yields

$$\begin{aligned}\psi &= \sin^{-1}\frac{6370}{6370+2000} \\ &= \end{aligned}$$

and $\triangle\omega - 2.21$ sr.

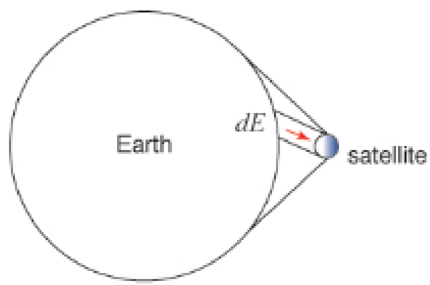


Fig. 4.37 Geometric setting for Exercises 4.30-4.34.

**4.31** If the Earth radiates as a black body at an equivalent blackbody temperature $T_E = 255$ K, calculate the radiative equilibrium temperature of the satellite when it is in the Earth's shadow.

Let $dQ$ be the amount of heat imparted to the satellite by the flux density $dE$ received within the infinitesimal element of solid angle $d\omega$. Then,

$$dQ = \pi r^2 I d\omega$$

where $r$ is the radius of the satellite and $I$ is the intensity of the isotropic blackbody radiation emitted by the Earth, which is given by

$$I = \frac{\sigma T_E^4}{\pi}$$

Hence,

$$dQ = r^2 \sigma T_E^4 d\omega$$

Integrating the above expression over the arc of solid angle subtended by the Earth, as computed in the previous exercise, yields the total energy absorbed by the satellite per unit time

$$Q = 2.21 r^2 \sigma T_E^4$$

For radiative equilibrium

$$Q = 2.21 r^2 \sigma T_E^4 = 4\pi r^2 \sigma T_s^4$$

where $T_s$ is the equivalent blackbody temperature of the satellite. Solving, we obtain

$$T_s = T_E \left( \frac{2.21}{4\pi} \right)^{1/4}$$

and substituting $T_E = 255$ K yields $T_s = 166$ K.

**4.32** Show that the approach in Exercise 4.5 in the text, when applied to the previous exercise, yields a temperature of

$$T_s = T_E \left[ \frac{1}{4} \left( \frac{6370}{8370} \right)^2 \right]^{1/4} = 158 \text{ K}$$

Applying the inverse square law, the flux density of Earth's radiation in the orbit of the satellite is

$$F = \sigma T_E \left( \frac{R_E}{d} \right)^2 \tag{1}$$

where $d$ is the distance between the center of the satellite and the center of the Earth. In analogy with Exercise 4.5, with the Earth's radiation treated as parallel beam, the radiation balance of the satellite is

$$\pi r^2 F_E = 4\pi r^2 \sigma T_s^4 \tag{2}$$

Combining (1) and (2) yields

$$T_s = T_E \left[ \left( \frac{1}{4} \right) \left( \frac{R_E}{d} \right)^2 \right]^{1/4} \tag{3}$$

This approach underestimates the temperature of the satellite because it treats the radiation that impinges on the satellite as parallel beam. The answer obtained in the previous exercise converges to this limiting value as $R_E/d \longrightarrow \infty$. In this limiting case, the angle $\psi$ in Exercise 4.30 becomes very small and converges to $\sin^{-1}(R_E/d) = R_E/d$. Using the half angle formula

$$\sin(x/2) = \sqrt{\frac{1 - \cos x}{2}}$$

the arc of solid angle

$$\triangle\omega = 2\pi(1 - \cos\psi)$$

subtended by the Earth in the "sky" of the satellite in Exercise 4.30 may be written as

$$\begin{aligned} \triangle\omega &= \pi \sin^2 2\psi \\ &= \pi \sin^2 (2R_E/d) \\ &= 4\pi (R_E/d)^2 \end{aligned}$$

From here on, we can proceed as in Method 1 of Exercise 4.21, replacing the (sun by the Earth) to show that the flux density of Earth radiation reaching the orbit of the satellite is given by the inverse square law

$$F_E = \sigma T_E^4 \left(R_E/d\right)^2$$

as assumed in Eq. (1). It follows that in this limiting case, the equivalent blackbody temperature of the satellite is the same regardless of whether it is computed using the equations in Exercise 4.5 or 4.30.

**4.33** Calculate the radiative equilibrium temperature of the satellite immediately after it emerges from the earth's shadow (i.e., when the satellite is sunlit but the earth, as viewed from the satellite, is still entirely in shadow).

*Answer* 289 K.

**4.34** The satellite has a mass of 100 kg, a radius of 1 m and a specific heat of $10^3$ J kg$^{-1}$ K$^{-1}$ . Calculate the rate at which the satellite heats up immediately after it (instantaneously) emerges from the earth's shadow.

*Answer* 0.043 °C s$^{-1}$.

**4.36** (a) Extend the proof in the previous exercise to the case in which absorptivity and emissivity are wavelength dependent. Let one of the walls be black, as in the previous exercise and let the other wall also be black, except within a very narrow wavelength range of width $\delta\lambda$, centered at $\lambda = \lambda_1$ where $a_{\lambda 1} < 1$. [**Hint:** Since blackbody radiation is isotropic, it follows the blackbody flux in the interval $\delta\lambda$ is $\pi B\left(\lambda_1, T\right)\delta\lambda$. Using this relationship, consider the energy balance as in the previous exercise and proceed to show that $\alpha_{\lambda 1} = \varepsilon_{\lambda 1}$.] (b) Indicate how this result could be extended to prove that

$$\varepsilon_\lambda = \alpha_\lambda$$

(a) If $\delta\lambda$ is of infinitesimal width, then in the region outside this band, the proof in the previous exercise should still be valid. Within the band

$$\begin{aligned} H &= \alpha_{\lambda 1}\pi B\left(\lambda_1, T\right)\delta\lambda - \varepsilon_{\lambda 1}\pi B\left(\lambda_1, T\right)\delta\lambda \\ &= \left(\alpha_{\lambda 1} - \varepsilon_{\lambda 1}\right)\pi B\left(\lambda_1, T\right)\delta\lambda \end{aligned}$$

Based on the same line of reasoning as in the previous exercise, it follows that $\alpha_{\lambda 1} = \varepsilon_{\lambda 1}$. (b) This proof could be extended by proving that $\varepsilon_{\lambda 2} = \alpha_{\lambda 2}$ within a second narrow wavelength range centered at $\lambda = \lambda_2$, and then within a third at $\lambda = \lambda_3$, and so on, to construct a spectrum in which $\varepsilon_\lambda$ and $\alpha_\lambda$ vary continuously, but are equal within any arbitrarily narrow wavelength range.

**4.37** Consider a closed spherical cavity in which the walls are opaque and all at the same temperature. The surfaces on the top hemisphere are black and the surfaces on the bottom hemisphere reflect all the incident radiation at all angles. Prove that in all directions $I_\lambda = B_\lambda$.

Downward directed radiation is blackbody radiation for which $I_\lambda = B_\lambda$ by definition. Since the bottom hemisphere is a perfect reflector, along each ray path, $I_{\lambda\Uparrow} = I_{\lambda\Downarrow}$ and since the lower hemisphere is at the same temperature as the upper hemisphere, $I_{\lambda\Uparrow} = I_{\lambda\Downarrow} = B_\lambda$.

**4.38** (a) Consider the situation described in Exercise 4.35, except that both plates are gray, one with absorptivity $\alpha_1$ and the other with absorptivity $\alpha_2$. Prove that

$$\frac{F_1'}{\alpha_1} = \frac{F_2'}{\alpha_2}$$

where $F_1'$ and $F_2'$ are the flux densities of the radiation emitted from the two plates. Make use of the fact that the two plates are in radiative equilibrium at the same temperature but do not make use of Kirchhoff's Law. [**Hint:** Consider the total flux densities $F_1$ from plate 1 to plate 2 and $F_2$ from plate 2 to plate 1. The problem can be worked without explicitly dealing with the multiple reflections between the plates.]

For each plate, the total radiation equals the emitted radiation plus the reflected radiation. For plate 1

$$F_1 = F_1' + F_2(1 - a_1)$$

and for plate 2

$$F_2 = F_2' + F_1(1 - a_2)$$

If the plates are in radiative equilibrium, $F_1 = F_2$, so we can write

$$F_1' + F_2(1 - a_1) = F_2' + F_1(1 - a_2)$$

or, since $F_1 = F_2$,

$$F_1' + F_1(1 - a_1) = F_2' + F_2(1 - a_2)$$

or

$$F_1' - a_1 F_1 = F_2' - a_2 F_2$$

which may be written as

$$\left(\frac{F_1'}{a_1} - F_1\right) a_1 = \left(\frac{F_2'}{a_2} - F_2\right) a_2$$

Since $F_1 = F_2$, but $a_1$ need not be equal to $a_2$, this relationship can be generally satisfied only if

$$\frac{F_1'}{\alpha_1} = \frac{F_2'}{\alpha_2}$$

**4.39** Consider the radiation balance of an atmosphere with a large number of isothermal layers, each of which is transparent to solar radiation and absorbs the fraction $\alpha$ of the longwave radiation incident on it from above or below. (a) Show that the flux density of the radiation emitted by the top-most layer is $\alpha F/(2-\alpha)$ where $F$ is the flux density of the planetary radiation emitted to space. By applying the Stefan-Boltzmann law (4.12) to an infinitesimally thin topmost layer, show that the radiative equilibrium temperature at the top of the atmosphere, sometimes referred to as the *skin temperature*, is given by

$$T^* = \left(\frac{1}{2}\right)^{1/4} T_E$$

(Were it not for the presence of stratospheric ozone, the temperature of the 20-80 km layer in the earth's atmosphere would be close to the skin temperature.)

Consider the radiation balance for the topmost layer of the atmosphere. Since this layer is isothermal and since its absorptivity $a$ equals its emissivity $\varepsilon$, it follows that it must emit flux density $F_1$ in both the upward and downward directions. Hence, it loses energy at a rate $2F_1$. The loss due to emission is balanced by the absorption of upwelling radiation from below. The upwelling radiation from below is $(F + F_1)$ and the fraction of it that is absorbed in the topmost layer is $a$. Hence

$$2F_1 = a(F + F_1) \tag{1}$$

Solving, we obtain

$$F_1 = \frac{\alpha F}{(2-\alpha)} \tag{2}$$

But from the Stefan Boltzmann law and Kirchhoff's law

$$F_1 = a\sigma T_1^4 \tag{3}$$

where $T_1$ is the equivalent blackbody temperature of the topmost layer. Combining Eqs. (2) and (3) yields

$$a\sigma T_1^4 = \frac{\alpha F}{(2-\alpha)}$$

In the limit, as $a \to 0$,

$$\sigma T_1^4 = \frac{F}{2} = \frac{\sigma T_E^4}{2}$$

Solving, we obtain

$$T_1 = (0.5)^{1/4} T_E$$

For the Earth;s atmosphere, $T_E = 255$ K and $T_1$, the so-called skin temperature, is 214 K, which is quite comparable to the mean temperature of the lower stratosphere.

**4.40** Consider an idealized aerosol consisting of spherical particles of radius $r$ with a refractive index of 1.5. Using Fig. 4.13, estimate the smallest radius for which the particles would impart a bluish cast to transmitted white light, as in the rarely observed "blue moon".

To impart a bluish (as opposed to a reddish) cast to transmitted light, the particles would need to exhibit a range of sizes for which the scattering efficiency $K_\lambda$ decreases with increasing values of the size parameter, the reverse of the situation for Rayleigh scattering. It is evident from Fig. 4.13, that the smallest range of values of $x$ for which $dK_\lambda/dx < 0$ occurs around

$$x = \frac{2\pi r}{\lambda} \sim 5$$

For visible light with $\lambda \sim 0.5\ \mu$m

$$\begin{aligned} 2\pi r &\sim 5 \times 0.5\ \mu\text{m}\ = 2.5\ \mu\text{m} \\ r &\sim 0.4\ \mu\text{m} \end{aligned}$$

**4.41** Consider an idealized cloud consisting of spherical droplets with a uniform radius of 20 $\mu$m and concentrations of 1 cm$^{-3}$. How long a path through such a cloud would be required to deplete a beam of visible radiation by a factor of $e$ due to scattering alone? (Assume that none of the scattered radiation is subsequently scattered back into the path of the beam.)

From Eq. (4.16),

$$dI_\lambda = -I_\lambda K_\lambda N \sigma ds$$

The size parameter $x = 2\pi r/\lambda = 2\pi \times 20\ \mu\text{m} \div 0.5\ \mu\text{m} > 50$. Hence, if absorption is assumed to be negligible, Fig. 4.13 yields a value of $K_\lambda = 2$. The number density $N = 1\ \text{cm}^{-3} = 10^6\ \text{m}^{-3}$ and the cross-sectional area $\sigma = \pi \times \left(20 \times 10^{-6}\right)^2 = 1.2 \times 10^{-9}$. Hence, the volume scattering coefficient $K_\lambda N \sigma = 2.4 \times 10^{-3}\ \text{m}^{-1}$. The incident radiation is depleted by a factor of $e$ over a path in which the optical depth $\tau_\lambda = \int K_\lambda N \sigma ds$ is equal to unity. If we assume that conditions are uniform along the ray path, the path length is given by

$$s = (K_\lambda N \sigma)^{-1} = \left(2.4 \times 10^{-3}\ \text{m}^{-1}\right)^{-1} \sim 400\ \text{m}$$

**4.43** Consider radiation with wavelength $\lambda$ and zero zenith angle passing through a gas with an volume absorption coefficient $k_\lambda$ of 0.01 m$^2$ kg$^{-1}$. What fraction of the beam is absorbed in passing through a layer containing 1 kg m$^{-2}$ of the gas? What mass of gas would the layer have to contain on order to absorb half the incident radiation?

The absorptivity of the layer is given by

$$A = 1 - e^{-\int k_\lambda u}$$

where $u$ is the density-weighted path length. For a layer containing 1 kg $\mathrm{m}^{-2}$ of gas, the optical depth $k_\lambda u = 0.01$. Since $e^x \cong 1 + x$ for $x << 1$, it follows that

$$A \cong 1 - (1 - 0.01) = 0.01 \text{ or } 1\%$$

For a layer thick enough to absorb exactly half the incident radiation,

$$T_\lambda = e^{-\tau_\lambda} = 0.5$$

Taking the natural logarithm yields

$$\tau_\lambda = \ln 0.5 = 0.694$$

Hence,

$$k_\lambda u = 0.694$$

and the mass per unit area along the path is

$$u = \frac{0.694}{k_\lambda} = 69.4 \text{ kg}$$

**4.45** For incident parallel beam solar radiation in an atmosphere in which $k_\lambda$ is independent of height, (a) show that optical depth is linearly proportional to pressure. (b) Show that the absorption *per unit mass* (and consequently the heating rate) is strongest, not at the level of unit optical depth but near the top of the atmosphere, where the incident radiation is virtually undepleted.

(a) If $k_\lambda$ and $r$ are assumed to be independent of height, optical depth is given by

$$\begin{aligned}
\tau_\lambda &= rk_\lambda \sec\theta \int_z^\infty \rho dz \\
&= \frac{rk_\lambda \sec\theta}{g} \int_z^\infty \rho g dz \\
&= \left(\frac{rk_\lambda \sec\theta}{g}\right) p
\end{aligned}$$

(b) The absorption per unit pass is equal to $-dI_\lambda/\rho$ where $\rho$ is the air density. In a hydrostatic atmosphere, the mass per unit height in a column. From Eq. (4.16)

$$dI_\lambda = I_\lambda \rho r k_\lambda \sec\theta dz$$

from which it follows that

$$\frac{dI_\lambda}{\rho} = -I_\lambda r k_\lambda \sec\theta dp/g$$

Provided that $r$, the density of the absorbing constitruent and $k_\lambda$ are not changing with height, it follows that the absorption per unit mass is strongest at the top of the atmosphere, where the monochromatic intensity of the incident radiation is undepleted.

**4.46** Consider a hypothetical planetary atmosphere comprised entirely of the gas in Exercise 4.43. The atmospheric pressure at the surface of the planet is 1000 hPa, the lapse-rate is isothermal, the scale height is 10 km and the gravitational acceleration is 10 m $s^{-2}$. Estimate the height and pressure of the level of unit normal optical depth.

At the level of unit normal optical depth

$$\tau_\lambda = k_\lambda \int_z^\infty \rho dz = 1$$

Hence,

$$\frac{k_\lambda}{g} \int_z^\infty \rho g dz = \frac{k_\lambda p}{g} = 1$$

It follows that

$$p = \frac{g}{k_\lambda} = \frac{10}{0.01} = 10^3 \text{ Pa}$$

From the hypsometric equation

$$\begin{aligned} z &= H \ln\left(\frac{p_0}{p}\right) \\ &= 10 \ln 100 \\ &= 46 \text{ km} \end{aligned}$$

**4.47** (a) What percentage of the incident monochromatic intensity with wavelength $\lambda$ and zero zenith angle is absorbed in passing through the layer of the atmosphere extending from an optical depth $\tau_\lambda = 0.2$ to $\tau_\lambda = 4.0$? (b) What percentage of the outgoing monochromatic intensity to space with wavelength $\lambda$ and zero zenith angle is emitted from the layer of the atmosphere extending from an optical depth $\tau_\lambda = 0.2$ to $\tau_\lambda = 4.0$? (c) In an isothermal atmosphere, through how many scale heights would the layer in (a) and (b) extend?

(a) The fraction of the incident radiation that is absorbed in the layer extending from an optical depth $\tau_\lambda = 0.2$ to $\tau_\lambda = 4.0$ is given by

$$e^{-0.2} - e^{-4.0} = 0.818 - 0.018 = 0.800$$

or 80%.

(b) Of the outgoing monochromatic intensity to space with wavelength $\lambda$ and zero zenith angle, the fraction $e^{-0.2}$ is emitted from the top of the layer without absorption, and the fraction $e^{-4.0}$ is emitted from the layer below without absorption. Hence, the fraction emitted from the layer is as given in (a).

(c) At the top of the layer

$$\tau_\lambda = k_\lambda \int_z^\infty \rho dz = 0.2$$

Hence,

$$\frac{k_\lambda}{g} \int_z^\infty \rho g dz = \frac{k_\lambda p_T}{g} = 0.2$$

and

$$p_T = 0.2 \frac{g}{k_p}$$

Similarly, at the bottom of the layer,

$$p_B = 4.0 \frac{g}{k_p}$$

The depth of the layer is

$$\begin{aligned}
\triangle z &= H \ln\left(\frac{p_B}{p_T}\right) \\
&= H \ln\left(\frac{4.0}{0.2}\right) \\
&= 3.00H
\end{aligned}$$

**4.48** For the atmosphere in Exercise 4.46, estimate the levels and pressures of unit (slant path) optical depth for downward parallel beam radiation with zenith angles of 30° and 60°.

At the level of unit optical depth

$$\tau_\lambda = k_\lambda \sec\theta \int_z^\infty \rho dz = 1$$

Hence,

$$p = \frac{g}{k_\lambda \sec\theta} = 0.866 \text{ and } 0.500 \times 10^3 \text{ Pa}$$

From the hypsometric equation

$$\begin{aligned}
z &= H \ln\left(\frac{p_0}{p}\right) \\
&= 10 \ln 115 \text{ and } 10 \ln 200 \\
&= 47.4 \text{ and } 53.0 \text{ km}
\end{aligned}$$

**4.49** Prove that the optical thickness of a layer is equal to minus the natural logarithm of the transmissivity of the layer.

By definition

$$T_\lambda = e^{-\tau_\lambda}$$

Taking the natural log of both sides yields

$$\tau_\lambda = -\ln T_\lambda$$

**4.50** Prove that the fraction of the flux density of overhead solar radiation that is backscattered to space in its first encounter with a particle in the atmosphere is given by

$$b = \frac{1-g}{2}$$

where $g$ is the asymmetry factor defined in (4.35). [**Hint**: The intensity of the scattered radiation must be integrated over zenith angle.]

**4.52** Prove that the ratio of the flux absorptivity to the intensity absorptivity at zero zenith angle approaches a limiting value of 2 as the optical thickness of a layer approaches zero.

The flux transmissivity, as defined in (4.45) is given by

$$T_\nu^f = 2\int_0^1 e^{-\tau_\nu/\mu}\mu d\mu$$

where $\mu = \cos\theta$. In the limit as $x \to 0$, $e^x \cong 1 + x$. Hence, as the optical thickness approaches zero,

$$\begin{aligned} T_\nu^f &= 2\int_0^1 \left(1 - \frac{\tau_\nu}{\mu}\right)\mu d\mu \\ &= 2\int_0^1 \mu d\mu - 2\int_0^1 \tau_\nu d\mu \\ &= 1 - 2\tau_\nu \end{aligned}$$

The corresponding flux absorptivity is given by

$$\alpha_\nu^f = 1 - T_\nu^f = 2\tau_\nu$$

and the intensity absorptivity is given by

$$\begin{aligned} \alpha_\nu &= 1 - T_\nu \\ &= 1 - e^{\tau_\nu} \\ &\cong \tau_\nu \end{aligned}$$

Hence,

$$\alpha_\nu^f = 2\alpha_\nu$$

**4.54** A thin, isothermal layer of air in thermal equilibrium at temperature $T_0$ is perturbed about that equilibrium value (e.g., by absorption of a burst of ultraviolet radiation emitted by the sun during a short lived solar flare) by the temperature increment $\delta T$. Using the cooling to space approximation (4.56) show that

$$\delta \left( \frac{dT}{dt} \right)_\nu = -\alpha_\nu \delta T \tag{4.65}$$

where

$$\alpha_\nu = \frac{\pi k_\lambda r}{c_p} \frac{e^{-\tau_\nu/\overline{\mu}}}{\overline{\mu}} \left( \frac{dB_\nu}{dT} \right)_{T_0} \tag{4.66}$$

This formulation, in which cooling to space acts to bring the temperature back toward radiative equilibrium is known as *Newtonian cooling* or *radiative relaxation.* It is widely used in parameterizing the effects of longwave radiative transfer in the middle atmosphere.

The cooling to space approximation [Eq. 4.59)] is

$$\left( \frac{dT}{dt} \right)_\nu = -\frac{\pi}{c_p} k_\nu r B_\nu(z) \frac{e^{-\tau_\nu/\overline{\mu}}}{\overline{\mu}}$$

If the layer is in radiative equilibrium at temperature $T_{0,}$ then there must be a heat source that exactly balances $(dT/dt)_\nu$. Let us assume that this heat source remains fixed, while the temperature of the layer increases from $T_0$ to $T_0 + \delta T$. The only term in (4.59) that changes in response to the warming of the layer is $B_\nu$ which changes by the increment $(dB_\nu/dT)_{T_0} \delta T$ in accordance with Planck's law. Hence, the imbalance in the heating rate is given by

$$\delta \left( \frac{dT}{dt} \right)_\nu = -\frac{\pi k_\nu r}{c_p} \frac{e^{-\tau_\nu/\overline{\mu}}}{\overline{\mu}} \left( \frac{dB_\nu}{dT} \right)_{T_0} \delta T$$

which is equivalent to Eqs. (4.65) and (4.66).

**4.55** Prove that weighting function $w_i$ used in remote sensing, as defined in (4.59) can also be expressed as the vertical derivative of the transmissivity of the overlying layer.

By definition

$$T_i = e^{-\tau_i}$$

where

$$\tau_i = \int_z^\infty k_i \rho r dz$$

Differentiating with respect to $z$, we obtain

$$\begin{aligned} \frac{dT_i}{dz} &= e^{-\tau_i} d\tau_i \\ &= e^{-\tau_i} k_i \rho r dz \end{aligned}$$

Hence, the weighting function in Eq. (4.59) can be expressed as

$$w_i = \frac{dT_i}{dz}$$

**4.56** The annual mean surface air temperature ranges from roughly 23°C in the tropics to −25°C in the polar cap regions. On the basis of the Stefan-Boltzmann law, estimate the ratio of the flux density of the emitted long-wave radiation in the tropics to that in the polar cap region.

From the Stefan Boltzmann law,

$$\begin{aligned}
\frac{F_{\text{tropics}}}{F_{\text{polar}}} &= \frac{\sigma T^4_{\text{tropics}}}{\sigma T^4_{\text{polar}}} \\
&= \left(\frac{273+27}{273-23}\right)^4 \\
&= \left(\frac{300}{250}\right)^4 \\
&= 2.07
\end{aligned}$$

**4.57** Consider the simplified model of the short wave energy balance shown in Fig. 4.38. The model atmosphere consists of an upper layer with transmissivity $T_1$, a partial cloud cloud layer with fractional coverage $f_c$ and reflectivity in both directions $R_c$, and a lower layer with transmissivity $T_2$. The planet's surface has an average reflectivity $R_s$. Assume that no absorption takes place within the cloud layer and no scattering takes place except in the cloud layer.

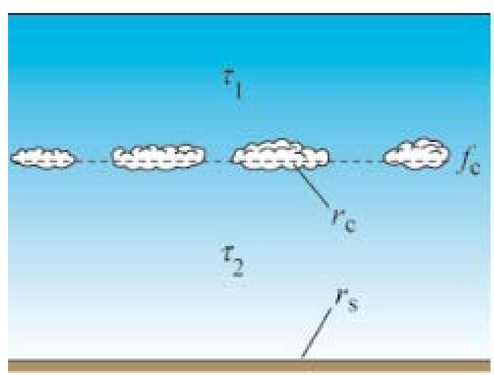


Fig. 4.38 Geometric setting for Exercise 4.57.

(a) Show that the total short wave radiation reaching the surface of the planet divided by the solar radiation incident on the top of the atmosphere is given by

$$F_s = \frac{[(1-f_c) + f_c\,(1-R_c)]T_1 T_2}{1 - T_2^2 f_c R_c R_s}$$

The fraction of the incident radiation reaching the surface of the planet, taking into account the radiation reflected upward from the surface and downward again from the cloud layer is

$$F_s = \{[(1 - f_c) + f_c (1 - R_c)]\, T_1 T_2\}[1 + T_2^2 f_c R_c R_s + \left(T_2^2 f_c R_c R_s\right)^2 + \left(T_2^2 f_c R_c R_s\right)^3 + .....]$$

Summing over the series, we obtain

$$F_s = \frac{[(1 - f_c) + f_c (1 - R_c)] T_1 T_2}{1 - T_2^2 f_c R_c R_s}$$

(b) Show that the planetary albedo is given by

$$A = f_c R_c T_1^2 + F_s R_s [(1 - f_c) + f_c (1 - R_c)]$$

The fraction of ther incident radiation that is reflected off the cloud tops is $f_c R_c T_1^2$ and the fraction refected off the surface of the planet is

(c) For the following values of model parameters calculate the planetary albedo: $f_c = 0.5$, $R_c = 0.5$, $T_1 = 0.95$, $T_2 = 0.90$, $R_s = 0.125$.

*Answer* 0.278

(d) Use the model to estimate the albedo of a cloud free and cloud covered earth.

*Answers* 0.09 and 0.48